Extremal Graph Theory - Fall '23 **Instructor: Asaf Shapira**

Home Assignment 4 **Due Date: TBA**

Please submit organized and well written solutions!

Problem 1. Show that there is no absolute constant C, so that for every $\epsilon > 0$ every graph has an ϵ-regular partition of order at most $(1/\epsilon)^C$.

Problem 2. Show that there is a K_7-free graph which has independence number $o(n)$ and more than $n^2/3$ edges. Also, show that if G is a K_7-free graph with $(\frac{1}{3} + \epsilon)n^2$ edges then it has independence number at least $\delta(\epsilon)n$.

Problem 3. Let G be a graph with $pn^2/2$ edges. We say that G satisfies property $\mathcal{P}$ if it contains at most $(p^4 + o(1))n^4$ copies of C_4. We say that G satisfies property $\mathcal{Q}$ if all but $o(n^2)$ pairs of vertices u, v satisfy $d(u, v) \le (p^2 + o(1))n$, where $d(u, v)$ is the co-degree of u, v, that is, the number of vertices that are adjacent to both u and v. Show that $\mathcal{P}$ and $\mathcal{Q}$ are equivalent.

Problem 4. The Erdős-Hajnal Theorem states that for every graph H there is a constant $c = c(H)$ so that every n-vertex graph that has no induced copy H, contains either a clique or an independent set of size $2^{c\sqrt{\log n}}$. Derive the Induced Ramsey Theorem from the Erdős-Hajnal Theorem.
Hint: Show that if $n \ge n_0(|H|)$ and G is an n-vertex graph with no clique or independent set of size $2 \log n$, then in any 2-coloring of $E(G)$ we can find a monochromatic *induced* copy of H.

Problem 5. Let G be a Red/Black-coloring of K_m where the largest Red clique has size $t-1$. Let us define a 2-coloring of the complete 3-uniform hypergraph on 2^m vertices as follows: we think of the vertices as strings in $\{0,1\}^m$ ordered lexicographically, with entries indexed by the vertices of G, and define $\delta(x, y)$ to be the largest entry where two vectors $x, y \in \{0,1\}^m$ differ. Then we color each triple of vertices $x < y < z$ with the color given to the edge $(\delta(x, y), \delta(y, z))$ in G.

- Show that this coloring has a red $K^3_{2^t}$.
- Show that there is a function f so that this coloring does not contain a red $K^3_{f(t)}$.
  Hint: Use Ramsey's Theorem.
- Suppose we change the coloring so that if $(\delta(x, y), \delta(y, z))$ is red then we color (x, y, z) red, but if $(\delta(x, y), \delta(y, z))$ is black then we color (x, y, z) black if $\delta(x, y) < \delta(y, z)$ and grey if $\delta(x, y) > \delta(y, z)$. Deduce from the previous item that there is a function $g(t)$, so that if G does not contain a red K_t or a black $K_{g(t)}$ then this coloring has no Red, Black or Grey K^3_t.
- Use the previous item to deduce that $R_3(t, t, t) \ge 2^{R_2(t, g(t))}$, where $R_2(s, t)$ is the usual Ramsey number for graphs.