Extremal Graph Theory

Instructor: Asaf Shapira
Scribed by Guy Rutenberg*

Fall 2016

Contents

*Submit comments to guyrutenberg@mail.tau.ac.il

1 First Lecture

All the graphs considered here are finite, simple, loop-free, etc. Unless otherwise mentioned, they will also be undirected. The following (very standard) notation is used throughout the notes. For a graph G we set $m = |E(G)|$ and $n = |V(G)|$. We use $N(x)$ to denote the neighborhood of a vertex x, and $d(x)$ to denote its degree (so $d(x) = |N(x)|$). We will use $\delta(G)$ to denote the smallest degree of a vertex in G, and $\Delta(G)$ to denote the largest degree. We use $e(G)$ to denote the number of edges of a graph G (so $e(G) = |E(G)|$). We will sometimes use $e(S)$ to denote the number of edges inside a vertex set S which is the subset of the vertex set of some graph. Similarly $e(A, B)$ will denote the number of edges connecting a vertex in A to a vertex in B. The complete graph on n vertices is denoted K_n, while the complete r-partite graph with partitions of sizes $n_1, n_2, \ldots, n_r$ is denoted $K_{n_1,\ldots,n_r}$.

We will frequently use *Jensen's inequality*, which states that if f is a convex function then for every $x_1, ..., x_n$ and $\alpha_1, ..., \alpha_n$ with $0 \le \alpha_i \le 1$ and $\sum_i \alpha_i = 1$ we have

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \le \sum_{i=1}^{n} \alpha_i f(x_i)$$

Exercise. Prove Jensen's Inequality (hint: use induction on n).

1.1 Mantel's Theorem

Theorem 1.1 (Mantel)**.** *If G is a triangle-free graph, then $m \leq \left\lfloor \frac{n^2}{4} \right\rfloor$. Furthermore, the only triangle-free graph with $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges is the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.*

Proof (First): It is easy to verify the proposition for $n = 3$ and $n = 4$, so assume $n > 4$ and note that it is enough to prove by induction that $m \leq n^2/4$. Let (x, y) be an edge in G. Clearly x and y cannot have a common neighbor (otherwise the graph will have a triangle), hence $d(x) + d(y) \leq n$. By removing x and y, we get a triangle-free graph $H = G - \{x, y\}$, which has, by induction, at most $\frac{(n-2)^2}{4}$ edges. Thus

$$m \leq \frac{(n-2)^2}{4} + n - 1 = \frac{n^2}{4} \ .$$

Now, if $m = \left\lfloor \frac{n^2}{4} \right\rfloor$, then all the above inequalities must be equalities. In particular, we must have $d(x) + d(y) = n$. It is then easy to see that this implies that $N(x) \cup N(y) = V(G)$. Since G is triangle-free, we know that $N(x)$ and $N(y)$ are both independent sets, implying that G is bipartite (with bipartition $N(x), N(y)$). Finally, note that the only bipartite graph on n vertices with $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges is $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. $\square$

Exercise. Prove Mantel's Theorem using induction on n, but remove only a single vertex each time.

Proof (Second): As G is triangle-free we infer (as in the previous proof) that $d(x) + d(y) \leq n$ for every edge $xy \in E(G)$. Summing this inequality over all the edges we get

$$\sum_{xy \in E} (d(x) + d(y)) \leq mn \ .$$

On the other hand, by Jensen's inequality

$$\sum_{xy \in E} (d(x) + d(y)) = \sum_{x} d^2(x) \underset{\substack{\uparrow \\ \text{Jensen}}}{\geq} n \left(\frac{\sum d(x)}{n} \right)^2 \underset{\substack{\uparrow \\ \sum_x d(x) = 2m}}{=} n \frac{4m^2}{n^2} = 4 \frac{m^2}{n}$$

thus $4\frac{m^2}{n} \leq mn$, giving the desired bound. Furthermore if $m = \left\lfloor \frac{n^2}{4} \right\rfloor$, then all the above inequalities must be equalities. In particular, we must have $d(x) + d(y) = n$ for any edge $xy \in E(G)$. As in the previous proof, this means that G must be $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. $\square$

Proof (Third): Let A be an independent set of maximal size in G. Since G is triangle-free we know that $N(x)$ is an independent set for every x, implying that $\Delta(G) \leq |A|$ (where we use $\Delta(G)$ to denote the maximum degree of a vertex in G). Setting $B = V \backslash A$, we see that as A is an independent set, every edge touches B. Therefore,

$$m \leq \sum_{x \in B} d(x) \leq |B| \Delta(G) \leq |B||A| \leq \left(\frac{|A| + |B|}{2} \right)^2 = \frac{n^2}{4} \ ,$$

implying the desired bound on m. Finally, if $m = \left\lfloor \frac{n^2}{4} \right\rfloor$, then the above inequalities are equalities and so B must be an independent set, implying that G is bipartite. As in previous proofs, this means that G must be $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. $\square$

Proof (Fourth): Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ be a vector satisfying $\sum x_i = 1$ and $0 \le x_i \le 1$. Define $f(x) = \sum_{ij \in E} x_i x_j$. We first observe that taking $x = (\frac{1}{n}, \ldots, \frac{1}{n})$ gives an x for which $f(x) \ge \frac{m}{n^2}$. We now claim that $f(x) \le \frac{1}{4}$ for every x. Indeed, if $ij \notin E$ then shifting the weight assigned to x_i to x_j (assuming the weight of the neighbors of x_j is at least as large as the weight assigned to the neighbors of x_i) does not decrease the value of f. If we repeat this process we get that f is maximized when all the weight is concentrated on a clique. As G is K_3-free, the weight is concentrated on two vertices, hence $f(x) = x_i x_j \le \frac{1}{4}$. Combining the above two observations, we see that $\frac{m}{n^2} \le \frac{1}{4}$. □

1.2 Turán's Theorem

Theorem 1.2 (Turán's Theorem (weak version))**.** *If G is K_{r+1}-free then $m \le (1 - \frac{1}{r})\frac{n^2}{2}$.*

Definition. Turán's graph, denoted $T_r(n)$, is the complete r-partite graph on n vertices which is the result of partitioning n vertices into r almost equally sized partitions ($\lfloor n/r \rfloor, \lceil n/r \rceil$) and taking all edges connecting two different partition classes (note that if $n \le r$ then $T_r(n) = K_n$). Denote the number of edges in Turán's graph by $t_r(n) = |E(T_r(n))|$.

Fact. *Turán's graph $T_r(n)$, maximizes the number of edges among all r-partite graphs on n vertices, and it is the only graph which does so.*

Exercise. Prove the above assertions.

Theorem 1.3 (Turán's Theorem (strong version))**.** *If G is K_{r+1}-free then $m \le t_r(n)$. Furthermore, equality holds if and only if $G = T_r(n)$.*

Proof (First): We start with the weak version, and proceed by induction on n, noting that the assertion is trivial for $n \le r$ (why?). Assume G is maximal K_{r+1}-free graph, hence it has a K_r subgraph. We denote this copy of K_r by X, and the rest of G by Y. Since G is K_{r+1}-free, the number of edges from any vertex in Y to X is at most $r-1$, therefore the number of edges between X and Y is bounded by $(n-r)(r-1)$. By induction the number of edges in Y is bounded by $(1 - \frac{1}{r})\frac{(n-r)^2}{2}$, thus the number of edges in G is at most

$$\binom{r}{2} + (n-r)(r-1) + \left(1 - \frac{1}{r}\right)\frac{(n-r)^2}{2} = \left(1 - \frac{1}{r}\right)\frac{n^2}{2} \ ,$$

proving the weak statement.

A close look at the above proof gives the strong statement as well. We again proceed by induction on n, noting that both assertions are trivial for $n \le r$. We first observe that removing a single vertex from each of the clusters of $T_r(n)$ gives $T_r(n-r)$, implying that

$$\binom{r}{2} + (n-r)(r-1) + t_r(n-r) = t_r(n) \ .$$

This implies the first part of the strong version. As to the second part, note that if $m = t_r(n)$, then Y must contain $t_r(n-r)$ edges and induction then gives that G restricted to Y must be isomorphic to $T_r(n-r)$ and is in particular a complete r-partite graph. Furthermore, equality also means that every vertex of Y must be connected to exactly $r-1$ of the vertices of X. It is then easy to see that since G is K_{r+1}-free that this means that G is also r-partite (we can add each vertex x of X to the cluster of $T_r(n-r)$ that has the vertices that are connected to all the vertices of X besides x. Convince yourself that this is well-defined). Since $T_r(n)$ is the only r-partite graph with $t_r(n)$ edges, the proof is complete. □

Proof (Second): We will prove that if G is a K_{r+1}-free graph on n vertices with the maximum possible number of edges, then it must be a complete k-partite graph. Since G is assumed to be K_{r+1}-free this means that $k \leq r$, and so $m \leq t_r(n)$ and that equality can only hold if $G = T_r(n)$ (again, since $T_r(n)$ is the only r-partite graph with $t_r(n)$ edges).

Showing that G is a complete k-partite graph for some k is equivalent to showing that for any triple of vertices $x, y, z \in V$, if $xz \notin E$ and $yz \notin E$ then $xy \notin E$. So assume $xz \notin E$ and $yz \notin E$ but $xy \in E$. If $d(z) < d(x)$, we can replace z by a vertex x' which will be connected to all of $N(x)$ (and thus to y). The resulting graph will still be K_{r+1}-free and will contain more edges. Hence, we can assume that $d(z) \geq d(x)$ and similarly $d(z) \geq d(y)$. Replacing both x and y by two vertices z' and z'' connected to $N(z)$ (and thus not connected to each other or to z), it is easy to check that we get a graph that is still K_{r+1}-free and contains more edges than G which is again a contradiction. $\square$

Proof (Third): We will prove by induction on n that if G has $t_r(n)$ edges and G is K_{r+1}-free then $G = T_r(n)$ (convince yourself that this indeed proves the theorem). The claim is clearly true when $n \leq r$. Let x be a vertex of minimal degree. Then $\delta(T_r(n)) \geq d(x)$ (as $T_r(n)$ maximizes $\delta(G)$ in comparison to every other graph with $t_r(n)$ edges). We thus get

$$e(G\backslash\{x\}) \geq e(G) - \delta(T_r(n)) = t_r(n) - \delta(T_r(n)) = t_r(n-1) \ ,$$

where the last equality follows from observing that $T_r(n-1)$ is obtained from $T_r(n)$ by removing a vertex from a cluster of size $\lceil n/r \rceil$, which amounts to removing $\delta(T_r(n))$ edges. Since $G\backslash\{x\}$ is K_{r+1}-free, we get from induction that $G\backslash\{x\} = T_r(n-1)$, so in particular $G\backslash\{x\}$ is r-partite. Since G is K_{r+1}-free, x is connected to vertices in at most $r-1$ of the partition classes of $G\backslash\{x\}$ so G is also r-partite and so must be isomorphic to $T_r(n)$. $\square$

The problem of proving existence of independent sets is of course closely related to that of finding cliques as each independent set corresponds to a clique in the complement graph and vice versa.

Proposition 1.4. *For every graph G we have*

$$\alpha(G) \geq \sum_x \frac{1}{1 + d(x)}$$

where $\alpha(G)$ is the size of the largest independent set in G.

Proof: Given some arbitrary ordering σ of the vertices of G, say $x_1, \ldots, x_n$, let I_σ be a vertex set with the property that $x_i \in I$ and $x_i x_j \in E$ then $i < j$ (all the neighbors of x_i appear after it in the ordering). Obviously I_σ is an independent set. Pick a random ordering σ of the vertices and define $X_v = 1 \iff v \in I$. Then

$$\mathop{\mathbb{E}}_\sigma[|I_\sigma|] = \mathbb{E}\left[\sum_v X_v\right] = \sum_v \mathbb{E}[X_v] = \sum_v P[X_v = 1] = \sum_v \frac{1}{1 + d(v)} \ .$$

Thus there is an ordering σ for which $|I_\sigma| \geq \sum_x \frac{1}{1+d(x)}$, and in particularly $\alpha(G) \geq \sum_x \frac{1}{1+d(x)}$. $\square$

1.3 Ramsey's Theorem

Theorem 1.5 (Ramsey)**.** *For every t there exists $N = R(t)$ such that every 2-coloring of the edges of K_N has a monochromatic K_t subgraph.*

Proof (First): We will show that $R(t) \le 4^t$. Take a graph on 2^{2t} vertices, and pick an arbitrary vertex v. At least one of the colors, say black, appears on at least half of the edges incident with v. We now set $x_1 = v$ and color it black (if the popular color was white we color x_1 white). Now, remove v and vertices u so that vu is colored white and note that we are left with at least 2^{2t-1} vertices. We repeat this process with whatever vertices left, resulting in $x_2, x_3, \ldots$. After i iterations, there are at least 2^{2t-i} vertices left, hence the process will pick $2t-1$ vertices. At least t of them will have the same color and it is easy to see that the way we defined them guarantees that they will span a monochromatic clique of size t. □

Proof (Second): We prove by induction on $s+t$ that

$$R(s,t) \le \binom{s+t-2}{s-1} ,$$

where $R(s,t)$ is the generalization of $R(t)$ (we want either black K_t or white K_s). Base case: when $s = 2$ ($t = 2$ is symmetric) then $R(2,t) = t$ (either all the edges black and we get a black K_t, or we have a white K_2 which is a single white edge). So suppose we have established the required upper bound for both $R(s-1,t)$ and $R(s,t-1)$. We will now prove that

$$R(s,t) \le R(s-1,t) + R(s,t-1) . \tag{1.1}$$

Induction will then imply that

$$R(s,t) \le \binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} = \binom{s+t-2}{s-1} .$$

Take a colored graph on $R(s-1,t) + R(s,t-1)$ vertices. Pick an arbitrary vertex x. Denote by A the vertices connected to x by black edges and by B those connected to it by white edges. Either $|A| \ge R(s,t-1)$ or $|B| \ge R(s-1,t)$, w.l.o.g. assume the former. Then, by the induction hypothesis A has a white K_s or black K_{t-1} and together with x it forms a black K_t, and anyway we are done. Hence (1.1) holds, and we are done. □

Corollary 1.6.

$$R(t) = R(t,t) \le \binom{2t-2}{t-1} \le \binom{2t}{t} = \frac{(2t)!}{(t!)^2} \approx \frac{\sqrt{4\pi t}\left(\frac{2t}{e}\right)^{2t}}{\left[\sqrt{2\pi t}\left(\frac{t}{e}\right)^t\right]^2} = \frac{4^t}{\sqrt{\pi t}}$$

Remark. Stirling approximation states that $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

Theorem 1.7. *For every $t \ge 3$ we have $R(t) \ge 2^{t/2}$.*

Proof: We first claim that if $\binom{n}{t}2^{1-\binom{t}{2}} < 1$ then $R(t) \ge n$. We can prove this using a "probabilistic argument" by noting that if we take a random coloring of K_n then the probability to have a monochromatic K_t subgraph is at most $\binom{n}{t}2^{1-\binom{t}{2}}$ so if the above inequality holds then with positive probability there will be no monochromatic K_t so there must be at least one coloring with no monochromatic K_t. We can also prove this using an equivalent "counting argument", as follows; the number of coloring of the complete graph on n vertices with a monochromatic K_t is bounded by $\binom{n}{t} \cdot 2 \cdot 2^{\binom{n}{2}-\binom{t}{2}}$ so if the above inequality holds then this number is smaller than the total number of colorings which is $2^{\binom{n}{2}}$, hence there must be one coloring with no monochromatic K_t.

We now just need to work out the largest n for which the above inequality holds. Note that $\binom{n}{t} \leq \frac{n^t}{t!}$, so taking $n = 2^{t/2}$ gives

$$\binom{n}{t} \cdot 2^{1-\binom{t}{2}} \leq \frac{n^t \cdot 2 \cdot 2^{t/2}}{t! \cdot 2^{t^2/2}} = \frac{2^{t^2/2} 2^{1+t/2}}{t! 2^{t^2/2}} = \frac{2^{t/2+1}}{t!} < 1$$

for every $t \geq 3$. $\square$

2 Second Lecture

In the last lecture, we proved Turán's Theorem, which says that $ex(n, K_{r+1}) \sim (1 - \frac{1}{r})\frac{n^2}{2}$. Today we will generalize it to arbitrary graphs.

2.1 The Erdős-Stone-Simonovits Theorem

Theorem 2.1 (Erdős-Stone-Simonovits)**.**

$$ex(n, H) = \left(1 - \frac{1}{\chi(H) - 1} \pm o(1)\right) \frac{n^2}{2} .$$

In other words for every H, r and ε, such that $\chi(H) = r + 1$, there exists $n_0 = n_0(r, H, \varepsilon)$ such that for every $n > n_0$

$$\left(1 - \frac{1}{\chi(H) - 1} - \varepsilon\right) \frac{n^2}{2} \leq ex(n, H) \leq \left(1 - \frac{1}{\chi(H) - 1} + \varepsilon\right) \frac{n^2}{2}$$

The lower bound is very easy; the Turán graph $T_r(n)$ is r-colorable, hence it is H-free (recall that we assume $\chi(H) = r + 1$). It is now easy to check (do this) that

$$e(T_r(n)) = t_r(n) \geq \left(1 - \frac{1}{r} - o(1)\right) \frac{n^2}{2} ,$$

so we are just left with proving the upper bound. We note that it is clearly enough to prove the upper bound only for K^v_{r+1}(which is the v-blowup[1] of K_{r+1}) as if $\chi(H) = r + 1$, then H is a subgraph of K^v_{r+1} for $|v| = |V(H)|$. So our goal is now to prove the following

Theorem 2.2 (Erdős-Stone-Simonovits (upper bound))**.** *For every r, v and $\varepsilon > 0$, there exists $n_0 = n_0(r, \varepsilon, v)$ such that every graph on $n \geq n_0$ vertices with $m \geq (1 - \frac{1}{r} + \varepsilon)\frac{n^2}{2}$ edges contains K^v_{r+1}.*

It will be easier to prove Theorem 2.2 by assuming that the graph is not only dense "globally" but also "locally". This is achieved by the following claim.

Claim 2.3. *For every $\varepsilon > 0$ there is $n_2 = n_2(\varepsilon)$ so that for every $n \geq n_2$ if G has n vertices and at least $(1 - \frac{1}{r} + \varepsilon)\frac{n^2}{2}$ edges, then it has a subgraph G' on $n' \geq \sqrt{\varepsilon}n/4$ vertices satisfying $\delta(G') \geq (1 - \frac{1}{r} + \frac{\varepsilon}{2})n'$.*

Given the claim above, it will be sufficient to prove the following variant of Theorem 2.2.

Theorem 2.4. *For every r, v and $\varepsilon > 0$, there exists $n_1 = n_1(r, \varepsilon, v)$ such that every graph G on $n \geq n_1$ vertices satisfying $\delta(G) \geq (1 - \frac{1}{r} + \varepsilon)n$ contains K^v_{r+1}.*

[1]The v-blowup of a graph K is obtained by replacing every vertex x of K by an independent set I_x of size v, and replacing every edge (x, y) by a complete bipartite graph connecting I_x to I_y.

Let us first show how to derive Theorem 2.2 from Claim 2.3 and Theorem 2.4.

Proof (Theorem 2.2): By Claim 2.3 if we start with G on $n \geq n_2(\varepsilon)$ vertices we can pass to a subgraph G' on $n' \geq \sqrt{\varepsilon}n/4$ vertices for which $\delta(G') \geq (1 - \frac{1}{r} + \frac{\varepsilon}{2})n'$. In order to be able to use Theorem 2.4 on G' we pick n_0 such that $n' \geq \sqrt{\varepsilon}n_0/4 \geq n_1(r, \frac{\varepsilon}{2}, v)$, so we set $n_0(r, \varepsilon, v) = \max\{4n_1(r, \varepsilon/2, v)/\varepsilon, n_2(\varepsilon)\}$. $\square$

To complete the proof of Theorem 2.2 we now turn to prove Claim 2.3 and Theorem 2.2.

Proof (Claim 2.3): We prune the vertices of G until we get the appropriate subgraph. As long as G has a vertex with a degree $\leq (1 - \frac{1}{r} + \varepsilon/2)n'$ we remove it from G (n' is the number of vertices currently left). Assume the process stopped with n' vertices. Thus

$$
\begin{aligned}
m &\leq n\left(1 - \frac{1}{r} + \frac{\varepsilon}{2}\right) + (n-1)\left(1 - \frac{1}{r} + \frac{\varepsilon}{2}\right) + \cdots + (n'+1)\left(1 - \frac{1}{r} + \frac{\varepsilon}{2}\right) + \binom{n'}{2} \\
&= \left(1 - \frac{1}{r} + \frac{\varepsilon}{2}\right)\left(\frac{n+n'+1}{2}\right)(n-n') + \binom{n'}{2} \\
&\leq \left(1 - \frac{1}{r} + \frac{\varepsilon}{2}\right)\left(\frac{n^2-(n')^2}{2}\right) + n + \frac{(n')^2}{2} \\
&\leq \left(1 - \frac{1}{r} + \frac{\varepsilon}{2}\right)\frac{n^2}{2} + (n')^2 + n\ .
\end{aligned}
$$

Since the claim's assumption is that $m \geq \left(1 - \frac{1}{r} + \varepsilon\right)\frac{n^2}{2}$, we can combine these lower/upper bounds on m to deduce that $(n')^2 \geq \frac{\varepsilon n^2}{4} - n$. If we add the assumption that $n \geq \frac{8}{\varepsilon}$ then the right hand side in this last inequality is at least $\frac{\varepsilon n^2}{8}$ implying that $n' \geq \sqrt{\varepsilon}n/4$, and the proof is complete (and we set $n_1(\varepsilon) = 8/\varepsilon$ as the function in the claim's statement). $\square$

Proof (Theorem 2.4): Assume the statement is true for $r-1$, namely that $n_1(r-1, \varepsilon, v)$ was defined for every v and $\varepsilon > 0$. Given v and $\varepsilon > 0$ we now wish to define $n_1(r, \varepsilon, v)$. We first ask that $n_1(r, \varepsilon, v) \geq n_1(r-1, \varepsilon, \bar{v})$, for a $\bar{v}$ that we will determine later. This means that if G has at least $n_1(r, \varepsilon, v)$ vertices and $\delta(G) \geq (1 - \frac{1}{r} + \varepsilon)n$ $(\geq (1 - \frac{1}{r-1} + \varepsilon)n)$ then G contains a copy of $K_r^{\bar{v}}$. Let $B_1, \ldots, B_r$ denote the r partition classes of this copy of $K_r^{\bar{v}}$. Our plan is to show that we can find in $U = V \backslash (B_1 \cup \cdots \cup B_r)$ a set $\bar{B}_0$ of v vertices so that for every $1 \leq i \leq r$, there is a set $\bar{B}_i \subsetneq B_i$ of size v, so that $\bar{B}_0, \bar{B}_1 \ldots, \bar{B}_r$ form a copy of K_r^v. Note that no matter how we pick $\bar{B}_1 \subseteq B_1, \ldots, \bar{B}_r \subseteq B_r$ these sets will form a complete r-partite graph, so we just need to make sure that the v vertices of $\bar{B}_0$ will be connected to all the vertices of $\bar{B}_1 \ldots, \bar{B}_r$.

Let $W \subseteq U$ contain all the vertices w such that w is connected to at least v vertices in each of the sets B_i. We claim that $|W| \geq \frac{\varepsilon n}{2}$. To prove this we bound $e(U, V \setminus U)$ from above and from below (recall that $V \backslash U = B_1 \cup \cdots \cup B_r$). Since $|B_1 \cup \cdots \cup B_r| = \bar{v}r$ we have as a lower bound

$$
e(U, V \setminus U) \geq r\bar{v}\delta(G) - \underset{\substack{\uparrow \\ \#\text{ edges in } U}}{(r\bar{v})^2} \geq r\bar{v}n\left(1 - \frac{1}{r} + \varepsilon\right) - (r\bar{v})^2 = (r-1)\bar{v}n + \varepsilon r\bar{v}n - (r\bar{v})^2\ .
$$

Since each vertex $u \in U \backslash W$ has some B_i where it has less than v neighbors, we have as an upper bound

$$
\begin{aligned}
e(U, V \setminus U) &\leq |W|r\bar{v} + (|U| - |W|)((v-1) + (r-1)\bar{v}) \\
&\leq |W|r\bar{v} + (n - |W|)(v + (r-1)\bar{v}) \\
&= |W|(\bar{v} - v) + vn + (r-1)\bar{v}n \\
&\leq |W|\bar{v} + vn + (r-1)\bar{v}n\ .
\end{aligned}
$$

Combining the above estimates for $e(U, V \setminus U)$ we infer that

$$|W|\bar{v} \geq \varepsilon r\bar{v}n - (r\bar{v})^2 - vn \ .$$

If we now add the requirement that $n \geq 4r\bar{v}/\varepsilon$ then we are guaranteed that $(r\bar{v})^2 \leq \varepsilon rn\bar{v}/4$, and if we assume that $\bar{v} \geq 4v/\varepsilon$ then we are guaranteed that $nv \leq \varepsilon rn\bar{v}/4$. So if both conditions hold then we get

$$|W|\bar{v} \geq \frac{\varepsilon rn\bar{v}}{2} \ ,$$

implying that $|W| \geq \frac{\varepsilon n}{2}$, as needed. So to make the above work we set $\bar{v} = 4v/\varepsilon$ and add the further assumption that $n \geq 4r\bar{v}/\varepsilon = 16rv/\varepsilon^2$. Up to now we assume that $n \geq \max\left\{n_1(r-1, \varepsilon, 4v/\varepsilon), 16rv/\varepsilon^2\right\}$.

Now, each of the vertices in W is connected to r vertices in each of the sets $B_1, \ldots, B_r$. There are $\binom{\bar{v}}{v}^r$ ways to choose r subsets $B'_1 \subseteq B_1, ..., B'_r \subseteq B_r$ of size v each. Therefore, there is a $W' \subseteq W$ of size at least $|W|/\binom{\bar{v}}{v}^r$ so that all the vertices in W' are connected to the same collection of subsets in $B_1, \ldots, B_r$. We can finally set this collection of subsets to be $\bar{B}_1 \ldots, \bar{B}_r$ and take $\bar{B}_0$ to be v of the vertices of W'. Of course we need to make sure that $|W'| \geq |W|/\binom{\bar{v}}{v}^r \geq \frac{\varepsilon n}{2}\binom{\bar{v}}{v}^r \geq v$ so we just need to further require that $n \geq \bar{v}^{rv}/\varepsilon$. We can finally set $n_1(r, \varepsilon, v) = \max\left\{n_1(r-1, \varepsilon, 4v/\varepsilon), 16rv/\varepsilon^2, (4v/\varepsilon)^{rv}/\varepsilon\right\}$. $\square$

2.2 The Zarankiewicz Problem for $K_{2,2}$

For $\chi(H) \geq 3$, the Erdős-Stone-Simonovits Theorem gives us a pretty good estimate for

$$ex(n, H) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right)\frac{n^2}{2} = \Theta(n^2) \ .$$

But when $\chi(H) = 2$ we get that $ex(n, H) = o(n^2)$, and wish to know exactly how "small" is this $o(n^2)$. We start by exploring complete bipartite graphs $K_{s,t}$ and specifically $K_{2,2}$.

Theorem 2.5.

$$ex(n, K_{2,2}) \leq \frac{n}{4}\left(1 + \sqrt{4n-3}\right) \sim \frac{n^{3/2}}{2}$$

Proof: Let us count the number of triples of vertices (x, y, z) so that x, y, z form a copy of $K_{1,2}$ with y as the middle vertex. Denote by $\#K_{1,2}$ the number of such triples. Then

$$\#K_{1,2} = \sum_v \binom{d(v)}{2} \underset{\substack{\uparrow \\ \text{Jensen}}}{\geq} n\binom{\frac{1}{n}\sum d(v)}{2} = n\binom{2m/n}{2} = \frac{2m^2}{n} - m \ .$$

If G is $K_{2,2}$-free then $\#K_{1,2} \leq \binom{n}{2}$, otherwise two of the $K_{1,2}$ share the same x, z coordinates forming a $K_{2,2}$. Combining the above two estimates on $\#K_{1,2}$ we infer that $\frac{2m^2}{n} - m \leq \binom{n}{2}$ or equivalently that $4m^2 - 2mn - n^3 - n^2 \leq 0$, implying the desired upper bound on m. $\square$

A lower bound (nearly) matching the one given in Theorem 2.5 is given in the next theorem.

Theorem 2.6.

$$ex(n, K_{2,2}) \geq \left(\frac{1}{2} - o(1)\right)n^{3/2}$$

Proof: We first show that for every prime p, we can construct a $K_{2,2}$-free graph on $n = p^2 - 1$ vertices with $m \geq (p^2-1)(p-1)/2 = \left(\frac{1}{2} - o(1)\right) n^{3/2}$ edges. We will then explain why this implies the lower bound for all n. So given a prime p we define a graph on $p^2 - 1$ vertices where each vertex is a pair $(a, b) \in F_p \times F_p$, $(a, b) \neq (0, 0)$. We "connect" vertex (a, b) to vertex (x, y) iff $ax + by = 1$ (over F_p).

Assume $v = (a, b) \neq (0, 0)$. Then, it is easy to check that in all cases (i.e. if v is either $(a, 0), (0, b)$ or (a, b) with $a, b \neq 0$), that there are exactly p solutions to $ax + by = 1$. This means that we always have $d(v) \geq p - 1$ (we omit the possible solution satisfying $x = a, y = b$ since we do not allow loops), implying that $m \geq (p^2 - 1)(p - 1)/2$ as needed.

To show that the graph is indeed $K_{2,2}$-free, take any $v = (a', b')$, $u = (a, b)$, $u \neq v$. Then the equations

$$ax + by = 1$$
$$a'x + b'y = 1$$

have at most one solution implying that v and u have at most one common neighbor, so the graph is indeed $K_{2,2}$-free.

The above construction works when n is a prime. But since it is known that for every integer n there exists a prime p satisfying $n \leq p \leq (1 + o(1))n$ the above lower bound applies to all values of n. □

2.3 The Zarankiewicz Problem for General $K_{s,t}$

Theorem 2.7. *One can extend the argument we used in order to prove the upper bound for $ex(n, K_{2,2})$* to obtain the following general bound*: For every $s \leq t$ we have*

$$ex(n, K_{s,t}) \leq \left(\frac{1}{2} + o(1)\right)(t-1)^{1/s} n^{2-\frac{1}{s}}$$

where $o(1) \to 0$ when $n \to \infty$.

Remark. Prove the above theorem. A slightly weaker bound (with $\frac{1}{2} + o(1)$ replaced by $e/2$ follows by using the inequalities $\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$.

We now give a general lower bound for $ex(n, K_{s,t})$.

Theorem 2.8. *For every $s \leq t$ we have*

$$ex(n, K_{s,t}) \geq \frac{1}{16} n^{2-\frac{s+t-2}{st-1}}$$

Proof: Consider the random graph $G(n, p)$. The expected number of edges and copies of $K_{s,t}$ are given by

$$\mathbb{E}[m] = p\binom{n}{2} > pn^2/4$$
$$\mathbb{E}[\#K_{s,t}] = \binom{n}{s}\binom{n}{t} p^{st} \leq n^{s+t} p^{st}$$

We now want to pick p so that we will have the relation

$$\mathbb{E}[\#K_{s,t}] \leq n^{s+t} p^{st} \leq \frac{1}{8} pn^2 \leq \frac{1}{2}\,\mathbb{E}[m] ,$$

so we set $p = 1/2n^{\frac{s+t-2}{st-1}}$. We then get that

$$\mathbb{E}\left[m - \#K_{s,t}\right] = \mathbb{E}[m] - \mathbb{E}[\#K_{s,t}] \geq \frac{1}{2}\,\mathbb{E}[m] \geq \frac{1}{8}pn^2 = \frac{1}{16}n^{2-\frac{s+t-2}{st-1}}\ .$$

So there exists a graph G in which $m - \#K_{s,t} \geq \frac{1}{16}n^{2-\frac{s+t-2}{st-1}}$, and by removing one edge from each $K_{s,t}$ we get a $K_{s,t}$-free graph with at least $\frac{1}{16}n^{2-\frac{s+t-2}{st-1}}$ edges. $\square$

3 Third Lecture

3.1 Applications of the Zarankiewicz Problem

Recall that last lecture we posed as an exercise to show that $ex(n, K_{s,t}) \leq 2(t-1)^{1/s}n^{2-1/s}$ for every $s \leq t$. We now give two applications of this bound.

Let S be a set of n points in the plane. How many pairs can be at distance 1? It is conjectured that the maximum number of such pairs is $n^{1+o(1)}$. Currently, the best known bound is $O(n^{4/3})$. Let us quickly observe a weaker upper bound of $O(n^{3/2})$. Given n points, we define a graph over these n points, where (u, v) is an edge iff $d(u, v) = 1$. It follows from elementary Geometry that this graph is $K_{2,3}$-free[2], hence by the above bound for $ex(n, K_{2,3})$ we get that the graph has $O(n^{3/2})$ edges, implying the desired bound on the number of pairs at distance 1.

We now move to a more complicated application.

Definition 3.1. Given a set of integers A, we denote by $A + A = \{a + a' \mid a, a' \in A\}$ its sumset.

We now ask how small can be a set $A \subset \mathbb{N}$ if $\left\{1, 2^2, 3^2, \ldots, m^2\right\} \subseteq A+A$? It is obvious that $|A| \geq \sqrt{m}$ (make sure you see this), but this bound can be improved as follows:

Theorem 3.2 (Erdős-Newman)**.** *If* $\left\{1, 2^2, 3^2, \ldots, m^2\right\} \subseteq A + A$ *then* $A \geq m^{2/3-o(1)}$.

Remark. The best known upper bound is a construction of a set of size $|A| \leq \frac{m}{\log^{\omega(1)}(m)}$, so there is still a huge gap between the best upper/lower bounds.

Proof: Suppose $\left\{1, 2^2, 3^2, \ldots, m^2\right\} \subseteq A + A$, and set $n = |A|$. Define a graph G whose vertices are the n members of A and where (a_1, a_2) is an edge iff $a_1 + a_2 = x^2$ for some $1 \leq x \leq m$. Note that the assumption on A implies that G has at least m edges. We will now show that G has no copy of $K_{2,t}$ where $t = n^{o(1)}$. The above bound on $ex(n, K_{s,t})$ then gives $m \leq |E(G)| \leq 2(n^{o(1)})^{1/2}n^{2-1/2} = n^{o(1)+3/2}$ implying that $n \geq m^{2/3-o(1)}$. Observe that showing that G has no copy of $K_{2,t}$ is equivalent to showing that every pair of vertices have at most t common neighbors. So our goal now is to show that every pair of vertices have $n^{o(1)}$ common neighbors.

Fix a pair of vertices a_1, a_2 and let b be one of their common neighbors. Then there are x, y satisfying $a_1 + b = x^2$ and $a_2 + b = y^2$, implying that $a_1 - a_2 = x^2 - y^2$. For each such common neighbor b we define a label $L_b = (x + y, x - y)$. Note that if b, b' are two distinct common neighbors of a_1, a_2 then $L_b \neq L_{b'}$ since given a label (p, q) we can recover x and y (by solving $x + y = p$ and $x - y = q$), and therefore also b itself. Now note that for any label (p, q) both p and q divide $a_1 - a_2$, hence the the number of common neighbors of a_1 and a_2 is at most $(D(|a_1 - a_2|))^2$ where $D(x)$ denotes the number of integers that divide x. Hence all that is left is to prove that $D(n) = n^{o(1)}$ (note that we can assume that $A \subseteq [m^2]$, so the difference between any pair of integers is at m^2, and $m^2 \leq n$ by a previous observation).

Suppose $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ is the prime factorization of n. Then $D(n) = \prod_{i=1}^{r}(1 + \alpha_i)$ and hence we need to prove that for every $\varepsilon > 0$ and large enough n, we have $\sum_{i=1}^{r} \log(1 + a_i) \leq \varepsilon \log n$ or equivalently that

[2] There is at most one circle of radius r passing through any 3 points, and a point connecting to those three points must be the center of that circle

$$\frac{\sum_{i=1}^{r}\log(1+\alpha_i)}{\sum_{i=1}^{r}\alpha_i\log p_i} \leq \varepsilon\ .$$

Observe that $\frac{\log(1+\alpha_i)}{\alpha_i\log p_i} \leq \varepsilon/2$ if either $\alpha_i \geq n_0(\varepsilon)$ or if $p_i \geq n_1(\varepsilon)$. So suppose there is $i_0 = i_0(\varepsilon)$ so that $\log(1+\alpha_i) \leq \frac{\varepsilon}{2}\alpha_i\log p_i$ for all $i > i_0$. Examining the numerator, we have that for some $m(\varepsilon)$ we have

$$\begin{aligned}\sum_{i=1}^{r}\log(1+\alpha_i) &= \sum_{i\leq i_0}\log(1+\alpha_i) + \sum_{i>i_0}\log(1+\alpha_i)\\ &\leq m(\varepsilon) + \frac{\varepsilon}{2}\sum_{i}\alpha_i\log p_i\\ &\leq m(\varepsilon) + \frac{\varepsilon}{2}\log n\ .\end{aligned}$$

We now conclude that

$$\frac{\sum_{i=1}^{r}\log(1+\alpha_i)}{\sum_{i=1}^{r}\alpha_i\log p_i} \leq \frac{m(\varepsilon)+\frac{\varepsilon}{2}\log n}{\log n} \leq \frac{m(\varepsilon)}{\log n} + \frac{\varepsilon}{2} \leq \varepsilon\ ,$$

for all large enough n, thus completing the proof. $\square$

3.2 The Turán Problem for Trees

We will now try to estimate $ex(n,T)$ where T is a tree on $t+1$ vertices (we will always assume that $t \geq 1$). A simple lower bound can be achieved by taking a disjoint union of cliques of size t, implying that $ex(n,T) \geq \frac{n(t-1)}{2}$. The following was a well known conjecture.

Conjecture 3.3 (Erdős-Sos)**.** *The above lower bound, $ex(n,T) \geq \frac{n(t-1)}{2}$, is tight.*

Theorem 3.4 (Ajtai-Komlos-Simonovits-Szemeredi)**.** *The Erdős-Sos Conjecture is correct.*

We will not prove the above theorem (whose proof actually has not appeared yet), but rather focus on some special cases and some weaker bounds. We start with the following simple but very useful fact.

Claim 3.5. *If G has m edges, then G has a subgraph H with min-degree $\delta(H) > \frac{m}{n}$.*

Proof: If $\delta(G) > \frac{m}{n}$ we are done. Else, pick $v \in V$ where $d(v) \leq \frac{m}{n}$, and remove it from the graph. The new average degree is at least

$$\frac{\sum_{i=1}^{n}d(i) - \frac{2m}{n}}{n-1} = \frac{2m\left(1-\frac{1}{n}\right)}{n\left(1-\frac{1}{n}\right)} = \frac{2m}{n}$$

hence the average degree never decreases. Continuing in this way we eventually end up with the required subgraph. $\square$

We now prove a weak version of the Erdős-Sos Conjecture.

Claim 3.6. *For every tree T on $t+1$ vertices, we have $ex(n,T) < (t-1)n$.*

Proof: If G has $(t-1)n$ edges, then by the above claim it has a subgraph H with min-degree at least t. So all that is left is to prove that if $\delta(H) \geq t$ then H has a copy of every tree of size $t+1$. Recall that every tree has a leaf, hence we can order the vertices $v_1, v_2, \ldots, v_{t+1}$ such that every v_i is connected to exactly one of the vertices $v_1, \ldots, v_{i-1}$. So assuming we have already found in H a copy of the tree spanned by $v_1, \ldots, v_{i-1}$ we now wish to find a vertex to represent v_i and thus obtain a copy of the tree spanned by $v_1, \ldots, v_i$. Assume v_i is connected to v_j, $1 \leq j \leq i-1$, thus we need to find a neighbor of v_j that does not belong to $v_1, \ldots, v_{i-1}$. As $d(v_j) \geq t$ and v_j has at most $t-1$ neighbors among the vertices $v_1, \ldots, v_{i-1}$ we can find such a vertex which we will take to be v_i. □

Claim 3.7. *Every graph has a cycle/path of length at least at least $\delta(G)+1$.*

Proof: Take the longest path in the graph, $x_1, \ldots, x_m$. All the neighbors of x_m must be on that path. One of them, together with x_m, closes a cycle of length at least $d(x_m)+1 \geq \delta(G)+1$. □

Theorem 3.8 (Dirac)**.** *If $\delta(G) \geq \frac{n}{2}$ then G has a Hamiltonian cycle.*

Proof: Consider a path, $x_1, \ldots, x_m$, of maximum length. All the neighbors of x_1 and x_m are on that path as well. We claim that there exists $1 \leq i \leq m-1$ such that $(x_1, x_{i+1}) \in E$ and $(x_i, x_m) \in E$. At least $n/2$ of the vertices $x_1, \ldots, x_{m-1}$ (which are at most $n-1$ vertices) are connected to x_m, and at least $n/2$ x_i vertices such that x_{i+1} is connected to x_1. By the pigeon-hole-principle, there is a x_i which fulfills both conditions.

Now, $x_1, \ldots, x_i, x_m, \ldots, x_{i+1}, x_1$ is a cycle of length m. If $n = m$ we are done. Otherwise, it is enough to show a vertex outside the cycle which is connected to it (which forms a path of length $m+1$). This holds if G is connected, and this follows directly from $\delta(G) \geq \frac{n}{2}$ (as every two vertices are either connected or share a common neighbor). □

It is easy to see that the lower bound on the min-degree in Dirac's Theorem is tight. For example, we can take two copies of K_n sharing a single vertex.

The next claim shows that we can improve the previous bound of $\delta(G)+1$ on the length of the longest path in a graph if we further assume the graph to be connected (note that without this assumption the bound $\delta(G)+1$ is tight).

Claim 3.9. *If G is connected, then G has a path of length* $\min\{n, 2\delta(G)+1\}$.

Proof: The proof is very similar to the proof of Dirac's Theorem and is thus left as an exercise. It is also easy to see that the bound is tight. □

Theorem 3.10 (Erdős-Gallai)**.** *If G is P_{t+1}-free then $m \leq \frac{(t-1)n}{2}$. Furthermore, equality holds only when G is a disjoint union of K_t's.*

Proof: By induction on n. For $n \leq t$, this is trivial. If G is not connected, we can apply induction on each of the connected component of G.

Assume that G is connected. Our goal is then to show that $m < \frac{(t-1)n}{2}$. Since G is connected and $n > t$ we infer that G must be K_t-free. Also, if $\delta(G) \geq \frac{t}{2}$, then by Claim 3.9 G has a path of length $\min\{n, t+1\} \geq t+1$ a contradiction. So assume G has a vertex with degree $\leq \frac{t-1}{2}$. Removing this vertex, we get from induction (using the assumptions that G is K_t-free and P_{t+1}-free) that the number of edges in the new graph is strictly smaller than $\frac{(t-1)(n-1)}{2}$. As we removed a vertex with degree $\leq \frac{t-1}{2}$, we get that $m < \frac{t-1}{2} + \frac{(t-1)(n-1)}{2} = \frac{(t-1)n}{2}$. □

3.3 The Girth Problem and Moore's Bound

Definition 3.11. The girth of a graph G, denoted $g(G)$, is the length of the shortest cycle in G.

We are interested in how many edges are needed to guarantee that $g(G)$ is small. We start with a lower bound.

Theorem 3.12. *There exists a graph on n vertices and $c'n^{1+\frac{1}{2k-1}}$ edges satisfying $g(G) > 2k$.*

Proof: Let $p = \frac{cn^{1/(2k-1)}}{n}$, for a constant c that will be chosen later, and consider $G(n,p)$. Then the expected number of edges and copies of C_t are given by

$$\mathbb{E}[m] \approx \frac{n^2}{2}p = \frac{c}{2}n^{1+\frac{1}{2k-1}}$$
$$\mathbb{E}[\#C_t] \leq n^t p^t \leq c^t n^{t/(2k-1)}$$

$$\mathbb{E}[\#C_3 + \#C_2 + \cdots + \#C_{2k}] = \sum_{t=3}^{2k} c^t n^{t/(2k-1)} = c^{2k} n^{1+\frac{1}{2k-1}} + O(n)$$

Therefore

$$\mathbb{E}[m - \#C_{\leq 2k}] = n^{1+\frac{1}{2k-1}} \left(\frac{c}{2} - c^{2k} - o(1)\right) \sim c'n^{1+\frac{1}{2k-1}}$$

for $c \leq \frac{1}{2}$. By taking a graph for which $m - \#C_{\leq 2k}$ is at least the expected value, and removing an edge from each short cycle, we get the desired graph. □

We now turn to prove an upper bound, which is usually referred to as the Moore Bound.

Theorem 3.13 (Moore's Bound)**.** *Every graph with at least $2n^{1+\frac{1}{k}}$ edges satisfies $g(G) \leq 2k$.*

Proof: We claim that every graph G on $m \geq 2n$ edges satisfies $g(G) \leq 2\log(n)/\log(m/n-1)$. To see this, we first use Claim 3.5 in order to find a subgraph G' of G on n' vertices with minimal degree $d > \frac{m}{n} \geq 2$. Set $g = g(G)$. Pick a vertex x in G' and grow a BFS tree of depth $g/2$ (assume g is even) with x as its root. Since $g(G') \geq g(G) = g$, we see that all vertices of this tree must be distinct, so

$$n' \geq 1 + d + d(d-1) + \cdots + d(d-1)^{g/2-1} \geq (d-1)^{g/2} ,$$

implying that

$$g \leq \frac{2\log n'}{\log(d-1)} \leq \frac{2\log n}{\log(m/n-1)} ,$$

as needed. Finally, if $m \geq 2n^{1+\frac{1}{k}}$ then the above bound implies that $g \leq \frac{2\log n}{\log(n^{1/k})} = 2k$. □

It is conjectured that the Moore bound gives the correct answer for the girth problem, namely, that there are graphs with $\Omega(n^{1+\frac{1}{k}})$ edges and no cycle of length at most $2k$.

3.4 Application of Moore's Bound to Graph Spanners

Given a graph G, we wish to find a sparse subgraph H which will satisfy the condition

$$\delta_G(u,v) \leq \delta_H(u,v) \leq (2k-1)\delta_G(u,v)$$

for every pair of vertices u, v. That is, we want to remove edges but to keep the distance between any pair of vertices close to the original distance. A graph satisfying the above condition is called a $(2k-1)$-spanner.

Theorem. *Every graph has a subgraph which is a $(2k-1)$-spanner with at most $n^{1+\frac{1}{k}}$ edges.*

Proof: Add the edges of G one by one where an edge is put into H only if the edge does not close a cycle of length $\leq 2k$. Then H has no cycle of length at most $2k$, hence by Moore's bound it has at most $n^{1+1/k}$ edges. It is easy to see that H is indeed a $(2k-1)$-spanner. □

Exercise. Show that if Moore's Bound gives the correct bound for the girth problem then there exist graphs, whose every $(2k-1)$-spanner has $\Omega(n^{1+\frac{1}{k}})$ edges.

4 Fourth Lecture

4.1 The Turán Problem for Long Cycles

Definition 4.1. The circumference of a graph G, denoted $c(G)$, is the length of the longest cycle in G.

We now ask how many edges are needed to guarantee that $c(G) > t$. The following theorem gives an exact answer.

Theorem 4.2 (Erdős-Gallai)**.** *If G has more than $\frac{t(n-1)}{2}$ edges then $c(G) \geq t+1$. Furthermore, this bound is tight when $n = k(t-1)+1$ (consider k copies of K_t all sharing a single vertex).*

To prove the above theorem, we will need the following lemma.

Lemma 4.3. *If G is 2-connected, then G has a cycle of length at least $\min\{n, 2\delta(G)\}$ (tight when G consists of several K_t sharing two vertices).*

Proof (Theorem 4.2): By induction on n. Since there is nothing to prove when $n \leq t$ assume $n > t$. If the graph has a vertex x where $\deg(x) \leq t/2$, then

$$e(G - \{x\}) \geq e(G) - \deg(x) > \frac{t(n-1)}{2} - \frac{t}{2} \geq \frac{t(n-2)}{2}$$

and by the induction hypothesis, the statement holds.

Otherwise, $\delta(G) \geq \frac{t+1}{2}$. If G is 2-connected, then by the lemma G has a cycle of length at least $\min\{n, t+1\} \geq t+1$. If G is not 2-connected, then it has a cut-vertex x which separates it into two otherwise disjoint components G_1 and G_2 which have n_1 and n_2 vertices. As G is assumed to contain more than $\frac{t(n-1)}{2} = \frac{t(n_1-1)}{2} + \frac{t(n_2-1)}{2}$ edges, we can apply the induction to either G_1 or G_2 (verify this!), completing the proof. □

We now turn to prove the lemma we used.

Proof (Lemma 4.3): Let $P = x_0, \ldots, x_m$ be the longest path in G. There are several possible case:

1. If there is a vertex x_i, where $0 < i < m$ such as x_0 is connected to x_i and x_m to x_{i-1}, then we can turn this path into a cycle (as in the proof of Dirac's Theorem). Furthermore, since G is assumed to be connected, this cycle must be a Hamilton cycle, as otherwise some vertex of the cycle has to be connected to a vertex not belonging to it, which gives a longer path.

2. If there is no vertex x_i as above, then assume there is $j < i-1$ such that x_0 is connected to x_i and x_m to x_j and take i, j such as $|i-j|$ is minimal (see Figure 4.1). Consider the cycle C that starts with x_0 goes to x_j then jumps to x_m, then goes to x_i and then jumps back to x_0. We claim that this cycle is of length at least $2\delta(G)$. By definition, C has x_m and all its neighbors (as P has all of

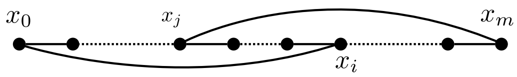


Figure 4.1: Cycle $C = x_0 \dots x_j x_m \dots x_i x_0$

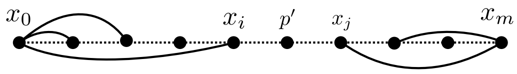


Figure 4.2: Third case in the proof of the lemma. The illustration assumes that there is no common neighbor, but this is not necessary for the proof.

x_m's neighbors, and none of them is between i and j), which are at least $\delta(G)$ vertices. Hence C has at least $\delta(G)+1$ vertices. Furthermore, if x_ℓ with $\ell \neq i$ is a neighbor of x_0 then C contains $x_{\ell-1}$ (as either $\ell > i$ or $i < j$). As we assume that the first case does not hold, $x_{\ell-1}$ is not a neighbor of x_m, hence C contains an additional set of $\delta(G)-1$ vertices. Altogether $|C| \geq 2\delta(G)$ as needed.

3. None of the neighbors of x_0 appears on P after a neighbors of x_m (see Figure 4.2). That is, if x_i is the furthest neighbor of x_0 and x_j is the furthest neighbour of x_m, then either $x_i = x_j$ or we have the situation in Figure 4.2. Let p' be the portion of P connecting x_i to x_j. Let A be the set consisting of x_0 and all its neighbors and B the set consisting of x_m and all its neighbors. As G is 2-connected there are two vertex disjoint paths Q_1, Q_2 which connect A and B. It is easy to see that p' allows us to assume wlog that one of the paths Q_1 or Q_2 starts at x_i. Indeed, we can start walking on p' from x_i till we either hit x_j (in which case we found a new path from A to B starting at x_i) or one of the paths Q_1 or Q_2 (in which case we can replace a portion of one of the paths with a portion of p'). By a similar argument we can now further assume that one of the paths starts at x_j (it might be the same one that starts at x_i). It is now easy to see that these two assumptions imply that we can construct a cycle covering $A \cup B$, (see Figure 4.3) which has length at least

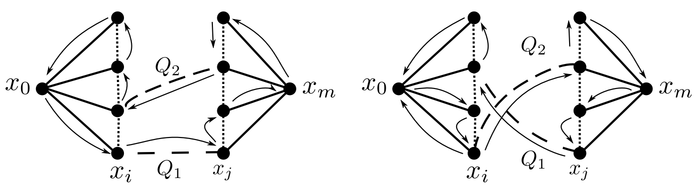
$$\deg(x_0) + 1 + \deg(x_m) + 1 \underset{\substack{\uparrow \\ \text{if } x_i = x_j}}{-1} \geq 2\delta(G)$$

These are all the possible cases, thus proving the lemma. □

4.2 Pancyclic Graphs and Bondy's Theorem

Definition 4.4. A graph is pancyclic if it contains a cycle of every length $3 \leq l \leq n$.

Theorem 4.5 (Bondy)**.** *If G is a Hamiltonian graph and $m \geq \frac{n^2}{4}$ then it is pancyclic unless $m = \frac{n^2}{4}$ and $G = K_{\frac{n}{2},\frac{n}{2}}$. [Bondy, 1971]*



(a) Q_1 has both x_i and x_j (b) Q_2 has x_i and Q_1 has x_j

Figure 4.3: Third case in the proof of the lemma. Dotted and dashed lines denote disjoint paths.

Corollary. *If $m > \frac{n^2}{4}$ and G is Hamiltoninan then G is pancyclic.*

Corollary. *If $\delta(G) > \frac{n}{2}$ then G is pancyclic.*

Proof (Bondy): Assume G has a cycle of length $n-1$, and let $x \notin C_{n-1}$. If $d(x) \leq \frac{n-1}{2}$ then in the induced subgraph on the cycle there are at least $\frac{n^2}{4} - \frac{n-1}{2} > \frac{(n-1)^2}{4}$ edges. By induction, we get that the graph induced by this C_{n-1} has a cycle of every length $\leq n-1$ and we are done. Assume then that $d(x) > \frac{n}{2}$. By the pigeonhole principle we get that for every $3 \leq l \leq n-2$ there are two neighbors of x that together with it form a cycle of length l (make sure you see this point) and we are done again.

So assume that G has no cycle of length $n-1$. We need to prove that $G = K_{\frac{n}{2},\frac{n}{2}}$. Note that for every pair of adjacent vertices x_i and x_{i-1} on the Hamilton cycle, and for every other vertex x_j, the graph can contain at most one of the edges (x_{i-1}, x_j) and (x_i, x_{j+2}), as otherwise we would get a cycle of length $n-1$. Therefore, for every i we have $d(x_i) + d(x_{i-1}) \leq n$ implying that

$$4m = \sum_{i=1}^{n} d(x_i) + d(x_{i-1}) \leq n^2 \ ,$$

implying that $m \leq \frac{n^2}{4}$. But we assume that $m \geq \frac{n^2}{4}$ so $m = \frac{n^2}{4}$, hence m must be even and:

($*$) for every i, j the graph G contains exactly one of the edges (x_{i-1}, x_j) or (x_i, x_{j+2}).

We are left with showing that G is bipartite (and then from $m = \frac{n^2}{4}$ it follows that $G = K_{\frac{n}{2},\frac{n}{2}}$). We claim that the alternating coloring along the Hamilton cycle is a legal coloring. Indeed, we cannot have chords of length 2 (i.e. x_i connected to x_{i+2}) as that would create a cycle of length $n-1$ (this actually follows by invoking ($*$) with x_{i-1}, x_i and $x_j = x_i$). This, together with ($*$) implies that we have all chords of length 3. This, together with ($*$) implies that we have none of the chords of length 4. Continuing in this fashion we deduce that there are no even chords (and that there are all odd chords) so the coloring is legal. □

4.3 The Moon-Moser Inequalities

Theorem 4.6. *For every graph G we have*

$$N_3 \geq \frac{4N_2}{3}\left(\frac{N_2}{N_1} - \frac{N_1}{4}\right)$$

where N_i is the number of copies of K_i in G (so $N_1 = n$ and $N_2 = m$).

Proof: For an edge e denote by $d(e)$ the number of triangles that contain e. Then we have

$$\begin{aligned}
3N_3 &= \sum_e d(e) \geq \sum_{(x,y)=e} (d(x) + d(y) - n) \\
&= \sum_x d^2(x) - mn \geq n\left(\frac{\sum d(x)}{n}\right)^2 - mn \\
&= \frac{4m^2}{n} - mn \ ,
\end{aligned}$$

and rearranging terms gives the required lower bound on N_3. □

Note that the above theorem implies that if a graph has $\frac{n^2}{4} + 1$ edges then it contains at least $\frac{n}{3}$, triangles. So in particular this implies Mantel's Theorem. Note that the $\frac{n}{3}$ bound on the number of triangles in graphs with $\frac{n^2}{4}+1$ edges is weaker than the (tight) $\lfloor n/2 \rfloor$ bound you were asked to prove in the home assignments. However, the theorem here is more general since it gives a lower bound for N_3 for any $m > n^2$. One can generalize the above theorem to the case of K_s.

Theorem 4.7 (Moon-Moser)**.** *(Generalization of the previous theorem) If $N_{s-1} \neq 0$:*

$$N_{s+1} \geq \frac{s^2 N_s}{s^2-1}\left(\frac{N_s}{N_{s-1}} - \frac{n}{s^2}\right)$$

Instead of proving the above theorem, we will prove the following more general theorem that deals with copies of K_s in r-uniform hypergraphs.

Theorem 4.8 (de-Caen)**.** *Moon-Moser Theorem for r-uniform hypergraphs. If $N_{s-1} \neq 0$:*

$$N_{s+1} \geq \frac{s^2 N_s}{(s-r+1)(s+1)}\left(\frac{N_s}{N_{s-1}} - \frac{(r-1)(n-s)+s}{s^2}\right)$$

Proof: We try to imitate the proof of Theorem 4.6. In what follows e denotes some copy of K_s. We start with the equation

$$(s+1)N_{s+1} = \sum_e d(e) \ ,$$

where $d(e)$ is the number of copies of K_{s+1} contain the vertex set of e. For each copy e of K_s, denote $e_1, \dots, e_s$ the s copies of K_{s-1} contained in it, where e_i is the result of removing vertex i from e. We will shortly prove that

$$d(e) \geq \frac{\sum_i d(e_i) - (n-s)(r-1) - s}{s-r+1} \ ,$$

but let us first show how to finish the proof using the above inequality. Using e' to denote copies of K_{s-1} we get

$$\begin{aligned}
(s-r+1)(s+1)N_{s+1} &= (s-r+1)\sum_e d(e) \\
&\geq \sum_e \left(\sum_i d(e_i) - (n-s)(r-1) - s\right) \\
&= \sum_{e'} d^2(e') - N_s\left((n-s)(r-1)+s\right) \\
&\geq N_{s-1}\left(\frac{\sum_{e'} d(e')}{N_{s-1}}\right)^2 - N_s\left((n-s)(r-1)+s\right) \\
&= \frac{s^2 N_s^2}{N_{s-1}} - N_s\left((n-s)(r-1)+s\right)
\end{aligned}$$

and rearranging the terms gives the theorem.

And now to the proof of the claim. We count how many copies of K_s there are with $s-1$ vertices in e (recall that e contains s vertices) and 1 in $V \backslash e$. Denote that number by p. We first observe that

$$p = \sum_{i=1}^{s} (d(e_i) - 1) = \sum_{i=1}^{s} d(e_i) - s \ .$$

Next, we note that in an r-uniform hypergraph, if a vertex u does not form a copy of K_{s+1} with a set of s vertices e, then it does not form an edge together with some $(r-1)$-subset of e. Hence, u can only create a copy of K_s together with subsets of e of size $s-1$ which result from omitting one of these $r-1$ vertices. In other words, u forms a copy of K_s with at most $r-1$ of the subsets of e of size $s-1$. Now, there are $d(e)$ vertices that form a copy of $K_{s+1}$with e, so each such vertex forms a copy of K_s with the s subsets of e of size $s-1$. By the above observation, the $n-s-d(e)$ vertices that do not form a copy of K_{s+1}, form a copy of $K_s$with at most $r-1$ of the subsets of e of size $s-1$. So we have

$$p \leq s \cdot d(e) + (n-s-d(e))(r-1) \ ,$$

and combining the above two estimates for p proves the claim. □

We now want to prove that if a graph has $m > \left(1-\frac{1}{t}\right)\frac{n^2}{2}$ edges, then it does not contain only one copy of K_t, but actually many of them. To this end

Claim 4.9. *Suppose G has m edges, and x is such that $m = \left(1-\frac{1}{x}\right)\frac{n^2}{2}$. If $x > s$ then*

$$\frac{N_{s+1}}{N_s} \geq \frac{n(x-s)}{x(s+1)}$$

Proof: Induction on s. For $s=1$ by the definition of m we have

$$\frac{N_2}{N_1} \geq \frac{n(x-1)}{2x} \ .$$

Assuming the claim holds for s we will prove it for $s+1$.

$$\begin{aligned}
\frac{N_{s+2}}{N_{s+1}} &\underset{\substack{\uparrow \\ \text{Theorem 4.7}}}{\geq} \frac{(s+1)^2}{(s+1)^2-1}\left(\frac{N_{s+1}}{N_s} - \frac{n}{(s+1)^2}\right) \\
&\underset{\substack{\uparrow \\ \text{Induction}}}{\geq} \frac{(s+1)^2}{(s+1)^2-1}\left(\frac{n(x-s)}{x(s+1)} - \frac{n}{(s+1)^2}\right) \\
&= \frac{n(x-s-1)}{x(s+2)}
\end{aligned}$$

□

Corollary 4.10. *If $m > (1-\frac{1}{s})\frac{n^2}{2}$ then $m = \left(1-\frac{1}{x}\right)\frac{n^2}{2}$ with $x > s$, hence $N_3 > 0$ implying $N_4 > 0$,...,implying $N_{s+1} > 0$. This is the weak version of Turán's Theorem.*

We now prove a more refined version of Turán's Theorem.

Corollary. *If G has $m = \left(1-\frac{1}{x}\right)\frac{n^2}{2}$ edges and $x > s$ then G contains at least $\left(\frac{n}{x}\right)^{s+1}\binom{x}{s+1}$ copies of K_{s+1}. In particular, if $m \geq (1-\frac{1}{s}+c)\frac{n^2}{2}$ then G has at least $c(\frac{n}{s+1})^{s+1}$ copies of K_{s+1}[3].*

Proof: For the first assertion, we infer from Claim 4.9 that

$$\frac{N_{s+1}}{N_1} = \frac{N_{s+1}}{N_s}\cdot\frac{N_s}{N_{s-1}}\cdot\ldots\cdot\frac{N_2}{N_1} \geq \prod_{i=1}^{s}\frac{n(x-i)}{x(i+1)} = \left(\frac{n}{x}\right)^s\frac{(x-1)(x-2)\cdots(x-s)}{1\cdot 2\cdot\ldots\cdot(s+1)} \tag{4.1}$$

Hence, $N_{s+1} \geq \left(\frac{n}{x}\right)^{s+1}\binom{x}{s+1}$ as we needed to show. As to the second assertion, note that if $m = (1-\frac{1}{s}+c)\frac{n^2}{2}$ then $x > s+c$ so for small enough c we get $N_{s+1} \geq c\cdot\left(\frac{n}{s+1}\right)^{s+1}$. □

[3]Follows from problem 3 in home assignment 1.

5 Fifth Lecture

5.1 The Hypergraph Turán Problem

It is natural to consider the Turán Problem in the setting of r-uniform hypergraphs (r-graphs for short). Surprisingly, while we have rather exact results for the graph Turán problem, already for 3-graphs the problem becomes much harder. In fact, it becomes much harder even in the "first" non-trivial case, when we forbid K_4^3, which is the complete 3-graph on 4 vertices. Let $ex(n, K_4^3)$ denote the maximum number of edges a 3-graph can contain if it does not contain a copy of K_4^3. It is not hard to show that $ex(n, K_4^3) \geq (\frac{5}{9} - o(1))\binom{n}{3}$; to this end we do something very similar to what we did in the graph case. We take n vertices and partition them into 3 sets V_1, V_2, V_3 of almost equal size. We then take as edges all triples of vertices (x, y, z) if they are of the form $x \in V_1, y \in V_2, z \in V_3$, or of the form $x, y \in V_i, z \in V_{i+1}$, where $i \in [3]$ (and addition is modulo 3). Let's call this graph $T^{3,4}$. A well known conjecture is that $ex(n, K_4^3) \leq (\frac{5}{9} + o(1))\binom{n}{3}$. To date, the best known upper bound is $ex(n, K_4^3) \leq 0.561\binom{n}{3}$.

A possible explanation for the hardness of proving tight bounds for hypergraph Turán Problem is the following. Recall that most proofs we gave in the graph case actually proved not only a tight upper bound but that there is a unique graph (the Turán graph) which attains the maximum. As it turns out, the 3-graph $T^{3,4}$ we described above is not the unique K_4^3-free 3-graph with this many edges. In fact, for every n there are exponentially many non-isomorphic 3-graphs that are K_4^3-free and have the same number of edges as $T^{3,4}$. So if indeed $ex(n, K_4^3) = |E(T^{3,4})|$ then any proof would have to "avoid" proving that $T^{3,4}$ is the unique maximum.

Looking back at our previous proofs of the graph Turán Problem, we can see that the one using the Moon-Moser inequality (see Corollary 4.10) did not prove uniqueness of the graph maximizing the number of edges. As it turns out one can also use the Moon-Moser inequality for r-graphs (see Theorem 4.8), to prove the following general upper bound.

Theorem 5.1. *Let K_s^r denote the complete r-graph on s vertices. Then*

$$ex(n, K_s^r) \leq \left(1 - \frac{1}{\binom{s-1}{r-1}} + o(1)\right)\binom{n}{r}$$

Observe that given the discussion above, this upper bound is not tight already for $ex(n, K_4^3)$. The proof of this theorem proceeds along the lines of the proof of Claim 4.9, but is much more tedious so we will skip it. Instead we will prove the following general lower bound

Theorem 5.2. *Let K_s^r denote the complete r-graph on s vertices. Then*

$$ex(n, K_s^r) \geq \left(1 - \left(\frac{r-1}{s-1}\right)^{r-1} - o(1)\right)\binom{n}{r}$$

Proof: It will be easier to consider the complementary problem of constructing an r-graph with at most $\left(\left(\frac{r-1}{s-1}\right)^{r-1} + o(1)\right)\binom{n}{r}$ edges so that every set of s vertices contains an edge (taking the complement of such an r-graph then proves the theorem). So consider the following r-graph; partition the n vertices to $s-1$ almost equal sized sets $V_1, \ldots, V_{s-1}$. Let e be a set of r vertices. Then e is an edge iff there exists an index j so that

$$\sum_{i=0}^{k-1} |e \cap V_{j+i}| \geq k+1, \quad \forall\, 1 \leq k \leq r-1\ , \tag{5.1}$$

where the subscripts of V_{j+i} are taken modulo $s-1$. Note that showing that every set of s vertices contains an edge follows from the fact that if we place s balls into $s-1$ buckets $V_1, V_2, \ldots, V_{s-1}$ that are arranged on a cycle, then there is an index j so that for every $1 \leq k \leq s-1$ the buckets $V_j, V_{j+1}, \dots, V_{j+k-1}$ have together at least $k+1$ balls. We leave the solution of this "riddle" as an exercise.

So we are left with proving an upper bound on the number of edges. It will turn out to be easier to count the number of ordered r-tuples $(x_1, \dots, x_r)$ such that $\{x_1, \dots, x_r\}$ is an edge. Let K denote the number of such r-tuples. Since $K = r!|E|$ (we are counting each edge exactly $r!$ times), what we need to show is that

$$K \leq r! \left(\left(\frac{r-1}{s-1} \right)^{r-1} + o(1) \right) \binom{n}{r} \leq (1+o(1)) n^r \left(\frac{r-1}{s-1} \right)^{r-1} .$$

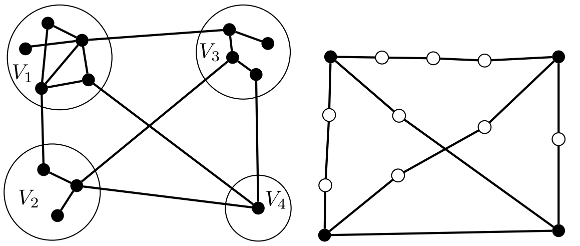
Let us assign to each r-tuple of vertices $(x_1, \dots, x_r)$, a "signature" (j, c), where $j \in \{1, \dots, s-1\}$ and $c = (c_1, \dots, c_r)$ is an r-tuple of non-negative integers satisfying $c_i = t$ iff $x_i \in V_{j+t}$. Observe that for every pair (j, c) there are $(1+o(1)) \left(\frac{n}{s-1} \right)^r$ ways to pick an r-tuple $(x_1, \dots, x_r)$ whose signature is (j, c). Note that the signature of an r-tuple $(x_1, \dots, x_r)$ determines whether it forms an edge, since it determines if it satisfies condition (5.1). Let us say that $(c_1, \dots, c_r)$ is legal if any r-tuple $(x_1, \dots, x_r)$ whose signature is (j, c) forms an edge [4]. Therefore, we are down to proving that there are at most $(r-1)^{r-1}$ ways to pick a legal $(c_1, \dots, c_r)$. Indeed, this will give that there are at most $(s-1)(r-1)^{r-1}$ signatures that define an edge[5], and since each signature defines $(1+o(1)) \left(\frac{n}{s-1} \right)^r$ ordered r-tuples $(x_1, \dots, x_r)$, this will give the required upper bound on K.

To prove the upper bound on the number of legal $(c_1, \dots, c_r)$, first note that condition (5.1) forces all $c_i \in \{0, 1, \dots, r-2\}$, as otherwise (5.1) will fail for $k = r-1$. This means that all legal c belong to $\{0, \dots, r-2\}^r$. So we need to prove that only $1/(r-1)$ of the strings in $c \in \{0, \dots, r-2\}^r$ define a legal c. To this end we will group these strings into groups of size $r-1$ and show that each group contains at most one legal c. Define $\text{shift}(c, \ell)$ to be the string c' satisfying $c'_i = c_i - \ell \pmod{r-1}$. So an r-tuple with signature $(j, \text{shift}(c, \ell))$ is obtained from an r-tuple with signature (j, c) by "rotating" the vertices "backwards" ℓ places along the $r-1$ clusters $V_j, \dots, V_{j+r-2}$. So $\text{shift}(c, r-1) = c$. Then each group contains $r-1$ strings of the form $\text{shift}(c, 0), \text{shift}(c, 1), \dots, \text{shift}(c, r-2)$. We claim that for any c at most one of the strings $\text{shift}(c, 0), \dots, \text{shift}(c, r-2)$ is legal. Indeed, if $\text{shift}(c, 0)$ is legal then c sends at least $\ell+1$ vertices to the clusters $V_j, \dots, V_{j+\ell-1}$, and if $\text{shift}(c, \ell)$ is also legal then c must also send at least $r-\ell$ vertices to the clusters $V_{j+\ell}, \dots, V_{j+r-2}$. The total is at least $r+1$ vertices, which is a contradiction. □

We note that the upper bound in the above argument is tight in the sense that the hypergraph indeed contains the stated number of edges. To see this we consider the two points where we obtained an upper bound. The first is when we claimed that each r-tuple $(x_1, \dots, x_r)$ that defines an edge has at least one index j for which condition (5.1) holds. It is not hard to see that there can never be two such indices, meaning that given the number of legal strings c, we were not "over counting" the number r-tuples $(x_1, \dots, x_r)$ that form an edge. To see this, suppose there are two such indices $1 \leq j < j' \leq s-1$; if $j' - j \geq r-1$ then there must be r vertices in the clusters $B_j, \dots, B_{j+r-2}$ and at least 2 more vertices in $B_{j'}$ for a total of at least $r+2$. A similar conclusion holds if the distance from j' to j is at least $r-1$, i.e. if $s-1-(j'-j) \geq r-1$. If none of the above conditions holds, then by a similar reasoning we must have at least $s+1 \geq r+2$ vertices. In any case we get a contradiction. A second point where the analysis is tight is where we claim that for any collection of $r-1$ strings $shift(c, 0), \dots, shift(c, r-2)$, at most 1 of them is legal. Again, it is not hard to see that one of them is always legal. The proof is very similar to the solution of the "riddle" we posed at the beginning of the proof.

[4] Here we are using the simple observation that for every c either all signatures (j, c) define an edge or none of them do.

[5] Note that we might actually be over counting the number of edges since an edge might (in theory) have two different signatures, that is, it might satisfy condition (5.1) for different j, j'. See the last paragraph for more details.



(a) K_4-minor (b) Topological K_4-minor

Figure 5.1

5.2 Extremal Problems Related to Graph Minors

We now turn to study a problem similar to the graph Turán problem, but instead of asking how many edged are needed to force the appearance of some graph H as a subgraph, we consider two relaxed notions of subgraphs, known as H-minor and topological H-minor. We start with the latter notion.

Definition 5.3. A graph G has a topological H-minor, where $H = (V_H, E_H)$ with $V_H = \{1, \ldots, h\}$, if G has h distinct vertices $u_1, ..., u_h$, and a collection of internally [6] vertex disjoint paths $\{P_{i,j} : (i, j) \in E_H\}$, so that $P_{i,j}$ connects u_i to u_j.

Definition 5.4. A graph G has an H-minor, where $H = (V_H, E_H)$ with $V_H = \{1, \ldots, h\}$, if G has h vertex disjoint connected subgraphs $U_1, ..., U_h$ and a collection of internally vertex disjoint paths $\{P_{i,j} : (i, j) \in E_H\}$, so that $P_{i,j}$ connects a vertex from U_i to a vertex of U_j.

Note that we could have defined the notion of a graph minor using single edges (i.e. paths of length 1) instead of the arbitrarily long paths we allowed in the definition. We just wanted to draw the similarity to the notion of topological minors. A more interesting equivalent notion of graph minor is the following.

Definition 5.5. A graph G has an H-minor if H can be obtained from G by a sequence of:

1. Vertex deletions
2. Edge deletions
3. Edge contractions (take an edge (x, y), and replace x, y with a single vertex connected to every neighbor of the vertices it replaced (removing loops and parallel edges).

One of the most interesting aspects of the notion of graph minor is its relation to the conjecture of Hadwiger which states the following.

Conjecture 5.6 (Hadwiger)**.** *If $\chi(G) \geq t$ then G has a K_t-minor.*

For some time the following conjecture was open.

Conjecture 5.7 (Hajos)**.** *If $\chi(G) \geq t$ then G has a topological K_t-minor.*

The following exercise guides you into disproving this conjecture Hajos's conjecture.

Exercise. Show that with high probability $G(n, 1/2)$ satisfies the following two properties:

[6] By this we mean that each path can meet other paths only at its first/last vertices. All the other vertices on the path are disjoint from $v_1, ..., v_h$and from the other paths.

1. It does not contain a topological K_t-minor with $t = 10\sqrt{n}$.

2. It is not $\frac{n}{2\log n}$-colorable.

Conclude that Hajos's conjecture is false. While you are at it, prove that with high probability, $G(n, 1/2)$ does contain a topological K_t-minor with $t = \sqrt{n}/10$.

Hadwiger conjecture, on the other hand, is still open. For $t = 2, 3$ it is trivial, for $t = 4$, it is an exercise. For $t = 5$ it turns out to be equivalent to the four color theorem. It is known to hold for $t = 6$ via a reduction to the four color theorem. For $t > 6$, the conjecture is open.

Let us get back to the extremal problems we are interested in. We first ask, how many edges are needed to guarantee that a graph has a K_t-minor. We will prove that surprisingly, a linear number of edges is sufficient.

Theorem 5.8. *Every graph with at least $2^{t-3}n$ edges has a K_t-minor.*

We also ask, how many edges are needed to guarantee that a graph has a topological K_t-minor. Again, we show that a linear number of edges is sufficient.

Theorem 5.9. *Every graph with at least $2^{\binom{t}{2}}n$ edges has a topological K_t-minor.*

The key fact that will enable us to prove these two theorems is Claim 5.11 stated below. For its proof we will need the following definition.

Definition 5.10. Given a graph G and a connected subgraph spanned by a vertex set S, denote by G/S the graph obtained from G by contracting S into a single vertex.

Claim 5.11. *Every graph G has a connected subgraph H, so that $\delta(G[N(H)]) > \frac{m}{n} - 1$, where $N(H)$ are the vertices that are adjacent to some vertex of H, and $G[N(H)]$ is the graph induced by these vertices.*

Proof: We describe a process for constructing H. Assume G is connected[7] and set $H_1 = \{v_0\}$ where v_0 is an arbitrary vertex of G. We now do the following; If $\delta\left(G[N(H_1)]\right) > \frac{m}{n} - 1$ we done. Otherwise, take a vertex $v_1 \in N(H_1)$, whose degree in $G[N(H_1)]$ is at most $\frac{m}{n} - 1$ and define $H_2 = H_1 + v_1$. We continue in this manner, where at iteration i we add to H_i a vertex $v_i \in N(H_i)$, whose degree in $G[N(H_i)]$ is at most $\frac{m}{n} - 1$ (if one exists, otherwise we are done) and define $H_{i+1} = H_i + v_i$. For what follows, observe that we always maintain that H_i spans a connected subgraph in G. Clearly when this process ends we get a set H satisfying the condition of the claim; the only thing that can go wrong is if $N(H) = \emptyset$. Since G is connected, this can only happen if $H_i = G$. To show that this cannot happen, we will show that if $d(G/H_i) \geq m/n$ then $d(G/H_{i+1}) \geq m/n$, where for a graph G we set $d(G) = m/n$. Since $G/H_1 = G$ we have $d(G/H_1) = d(G) = m/n$ implying that $d(G/H_i) \geq m/n$ so $H_i \neq G$.

Set n_i/m_i to be the number of vertices/edges in G/H_i, and n_{i+1}/m_{i+1} to be the number of vertices/edges in G/H_{i+1}. Then $n_{i+1} = n_i - 1$. As to m_{i+1} note that when adding v_i to H_i we lose the edge that connected v_i to the vertex resulting from contracting H_i as well as the at most $\frac{m}{n} - 1$ edges that connected v_i to its neighbors in $G[N(H_i)]$ (all these edges will be lost since H_i is already connected to them). Altogether we get $m_{i+1} \geq m_i - m/n$. Thus

$$\frac{m_{i+1}}{n_{i+1}} \geq \frac{m_i - m/n}{n_i - 1} \geq \frac{m}{n}\ ,$$

where in the second inequality we use the assumption that $m_i/n_i \geq m/n$. □

[7]Why is it fine to assume so? If G was not connected, at least one of its connected components must be at least as dense as G and we could operate on it instead of G.

The proof of Theorems 5.8 and 5.9 are now easy corollaries of the claim above.

Proof (Theorem 5.8): Induction on $t \geq 3$. For $t = 3$ it is trivial[8]. Assume now that G has $m = 2^{t-3}n$ edges where $t \geq 4$. Then from Claim 5.11 we get a subset H such that $\delta(G[N(H)]) > 2^{t-3} - 1$ that is $\delta(G[N(H)]) \geq 2^{t-3}$. Since $\delta(G[N(H)]) \geq 2^{t-3}$ we get that if $G[N(H)]$ has n' vertices then it has at least $2^{t-4}n'$ edges. By induction, $G[N(H)]$ has K_{t-1}-minor and together with H we get a K_t-minor. $\square$

Proof (Theorem 5.9): We prove by induction on k that $2^{t+k}n$ edges give a topological K-minor where K is a cycle of length t with k chords. If $k = 0$, then we have a graph with $2^t n$ edges. Claim 3.5 then gives a subgraph with minimal degree at least $2^t + 1$ and Claim 3.7 then a cycle of length $2^t + 2$ which is clearly a C_t-minor (any cycle of length at least t is a C_t minor). Given a graph with $2^{t+k}n$ edges we get (like in the previous proof) a connected subgraph H, so that $N(H)$ has average degree 2^{t+k}. By induction we can find in $N(H)$ a topological K'-minor where K' is a C_t plus $k - 1$ chords. We can now pick two vertices in the topological K'-minor that are not yet connected by a path, and connect them by a path that goes through H and avoids all other vertices/paths that belong to $N(H)$.

$\square$

It is of course natural to ask whether the constant 2^{t-3} in Theorem 5.8 can be improved? As it turns out $O(t\sqrt{\log t}n)$ edges are sufficient and necessary for forcing a K_t-minor. Proving that this many edges suffice is somewhat harder than the proof of Theorem 5.8, but the proof of the necessity of this many edges follows from the following exercise

Exercise 5.12. Show that with high probability, $G(n, 1/2)$ does not contain a K_t-minor with $t = 4n/\sqrt{\log n}$.

Taking an appropriate $n = n(t)$ one gets from the above exercise an n-vertex graph with average degree $\Omega(t\sqrt{\log t})$ and no K_t-minor. Taking disjoint copies of such a graph, gives arbitrarily large graphs with $\Omega(t\sqrt{\log t}n)$ edges and no K_t-minor. Let us note that with high probability $G(n, 1/2)$ is $n/2 \log n$-colorable and contains a K_t-minor with $t = n/2\sqrt{\log n}$, so as opposed to Hajos's Conjecture, $G(n, 1/2)$ satisfies Hadwiger's Conjecture.

As to Theorem 5.9 , it turns out that $O(t^2 n)$ are sufficient and necessary for forcing a K_t-minor. Again, the upper bound is hard while the lower bound follows from one of the items in Exercise 5.2 regarding $G(n, 1/2)$. Actually, as the following exercise suggests, one does not even need to consider random graphs.

Exercise 5.13. Show that $K_{t^2/10,t^2/10}$ does not contain a topological K_t-minor.

As in the case of K_t-minors, taking several disjoint copies of $K_{t^2/9,t^2/9}$ we get arbitrarily large graphs with $\Omega(t^2 n)$ edges and no topological K_t-minor.

5.3 Application of Topological K_t-Minors to Graph Linkage

Definition. A graph G is called k-linked if for every collection of k pairs of vertices $(s_1, t_1), \ldots, (s_k, t_k)$, with all $2k$ vertices being distinct, one can find in G a collection of k vertex-disjoint paths $P_1, \ldots, P_k$ such that P_i connects s_i to t_i.

Note that being k-linked is a stronger property than being k-connected, that is, every k-linked graph is also k-connected. It is not clear a-priori that any measure of connectivity guarantees that a graph is k-linked. The following theorem shows that this is actually the case.

Theorem 5.14 (Bollobas-Thomasson)**.** *If G is $22k$-connected then G is k-linked [Bollobas and Thomason, 1996].*

[8]The graph has at least n edges $\implies$ it must contain a cycle.

Exercise 5.15. Use the above theorem to show that for large enough C, every graph with Ct^2n edges contains a topological K_t-minor. You might find it useful to use Mader's Theorem that states that a graph with average degree $4k$ contains a k-connected subgraph.

We now prove a weaker version of Theorem 5.14.

Theorem 5.16. *If G is $2 \cdot 2^{\binom{3k}{2}}$-connected then G is k-linked.*

Proof: We prove a stronger statement, that if $h(t)$ is such that every graph with average degree $h(t)$ contains a topological K_t-minor, then every graph which is $(h(3k)+2k)$-connected is k-linked. Recall that we proved in Theorem 5.9 that $h(t) \leq 2^{\binom{t}{2}}$ thus giving the result in the statement. Note further, that if instead we plug in the fact that $h(t) = O(t^2)$ (which we did not prove) we get that every $O(k^2)$-connected graph is k-linked, which comes close to the linear bound of Theorem 5.14. Combining this with the exercise above we get that bounding the function $h(t)$ is essentially equivalent to proving Theorem 5.14.

Let us turn to proving the statement mentioned above. If G is $(h(3k)+2k)$-connected then $\delta(G) \geq h(3k)$ hence by definition of $h(t)$ we get that G contains a topological K_{3k}-minor, which we denote by K. Let U denote the $3k$ "hubs" of K. By Menger's theorem, G has $2k$ vertex disjoint paths $P_1, \ldots, P_{2k}$ from $s_1, t_1, \ldots, s_k, t_k$ to U. Pick a collection of $P_1, \ldots, P_{2k}$ as above so that it would minimize the number of edges *not* belonging to K. We now show how to construct the k paths connecting each of the pairs s_i, t_i.

Fix a pair s_i, t_i. Suppose the two paths connecting them to U end at the vertices x_i, y_i. Pick any one of the k vertices of U that are not an endpoint of $P_1, \ldots, P_{2k}$ and call it u_i. Let L be the path of K connecting u_i to x_i and let M be the path of K connecting u_i to y_i. Let ℓ be the vertex of L which is closest to u_i with the property that one of the paths P of $P_1, \ldots, P_{2k}$ intersect it. It is easy to see that the minimality property of $P_1, \ldots, P_{2k}$ implies that P must be the path connecting s_i to x_i. We can define m in a similar fashion and conclude that the path intersecting it must be the path connecting t_i to y_i. It is now easy to see that we can connect s_i to t_i "via" u_i. $\square$

6 Sixth Lecture

6.1 Introduction to Szemerédi's Regularity Lemma

What is the Regularity Lemma?

Definition 6.1. A bipartite graph (A, B) is called ε-regular if for every $A' \subseteq A$ and $B' \subseteq B$ such that $|A'| \geq \varepsilon|A|$ and $|B'| \geq \varepsilon|B|$, we have

$$|d(A', B') - d(A, B)| \leq \varepsilon$$

where $d(X, Y) = e(X, Y)/|X||Y|$ is the edge density between X and Y.

Note that an ε-regular bipartite graph has certain properties one expects to find[9] in a random bipartite graph of edge density d. So one can think of being ε-regularity as being ε-pseudo-random.

Exercise. Show that if $|X| = |Y|$ and $d(X, Y) \leq \varepsilon^3$ then (X, Y) is ε-regular.

Definition 6.2. A partition $P = V_1, ..., V_k$ of $V(G)$ is called an equipartition if the sizes all the partitions differ by 1 at most. The order of P is the number of sets in it (k above).

[9] To be more precise, we would have expected such behavior in a random graph even from subsets of size $\frac{\log n}{\varepsilon}$.

Definition 6.3. An equipartition $V_1, ..., V_k$ of $V(G)$ is called ε-regular if all but at most εk^2 of the pairs (V_i, V_j) form ε-regular graph.

Theorem 6.4 (Regularity Lemma)**.** *For every $\varepsilon > 0$ there exists $T = T(\varepsilon)$ such as every graph has an ε-regular equipartition of order k where $\frac{1}{\varepsilon} \leq k \leq T(\varepsilon)$ [Szemerédi, 1975].*

6.2 The Triangle-Removal Lemma and Roth's Theorem

We say that a graph G is ε-far from being triangle free if one has to remove at least εn^2 edges from G in order to make it triangle-free. We now consider the following problem; what is the relation between how far G is from being triangle-free and the number of triangles in G? One direction is very easy.

Exercise. Show that if G contains εn^3 triangles then G is ε-far from being triangle free.

The following famous lemma sates that the reverse implication also holds.

Theorem 6.5 (Triangle Removal Lemma)**.** *For every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such as if G is a ε-far from being triangle-free then G has at least δn^3 triangles [Ruzsa and Szemerédi, 1978].*

For the proof of the triangle-removal lemma we will use the following lemma. Note that we can think of its statement as saying that if a 3-partite graph is dense/regular enough then it contains approximately the same number of triangles one expects to find in a "truly" random 3-partite graph of the same density.

Lemma 6.6. *If A, B, C are vertex sets such as $d(A, B) = b, d(A, C) = c, d(B, C) = a$ and $a, b, c \geq 2\varepsilon$ and all three bipartite graphs are ε-regular then the graph (A, B, C) has at least*

$$(1 - 2\varepsilon)(a - \varepsilon)(b - \varepsilon)(c - \varepsilon)|A||B||C|$$

triangles.

Proof: Note that A has at most $\varepsilon|A|$ vertices whose number of neighbors in B is smaller than $(b-\varepsilon)|B|$. Otherwise, this set of vertices together with B would have given a pair of subsets contradicting the ε-regularity of (A, B). The same goes for A and C. All in all, all the vertices of A, apart from $\leq 2\varepsilon|A|$, have at least $(b-\varepsilon)|B|$ neighbors is B and $(c-\varepsilon)|C|$ neighbors in C. Let $v \in A$ be one of the "good" vertices. As $b, c \geq 2\varepsilon$ we have $|N_B(v)| \geq \varepsilon|B|$ and $|N_C(v)| \geq \varepsilon|C|$, so we can apply the regularity condition to deduce that the number of edge between $N_B(v)$ and $N_C(v)$ is at least

$$(a - \varepsilon)|N_B(v)||N_c(v)| \geq (a - \varepsilon)(b - \varepsilon)(c - \varepsilon)|B||C|$$

thus every "good" vertex takes part in at least that number of triangles. As at least $(1 - 2\varepsilon)|A|$ are "good" vertices we get that there are at least

$$(1 - 2\varepsilon)(a - \varepsilon)(b - \varepsilon)(c - \varepsilon)|A||B||C|$$

triangles. □

Proof (Triangle Removal Lemma): Given a graph G we apply the Regularity Lemma with $\varepsilon/4$. We get an $\varepsilon/4$-regular partition $V_1, \ldots, V_k$ with $\frac{4}{\varepsilon} \leq k \leq T\left(\frac{\varepsilon}{4}\right)$. We remove from the graph the following edges:

1. Edges that belong to one of the clusters V_i. There are at most $k\frac{(n/k)^2}{2} \leq \frac{n^2}{2k} \leq \frac{\varepsilon n^2}{8}$ such edges.
2. Edges between non ε-regular (V_i, V_j) pairs. There are most $\frac{\varepsilon}{4}k^2(n/k)^2 = \frac{\varepsilon}{4}n^2$ such edges.

3. Edges between (V_i, V_j) such that $d(V_i, V_j) \leq \frac{\varepsilon}{2}$. There are at most $\binom{k}{2}\frac{\varepsilon}{2}(n/k)^2 \leq \frac{\varepsilon}{4}n^2$ such edges.

As we removed at most $\frac{5}{8}\varepsilon n^2$ edges, the new graph, call it G', still has a triangle. From the type of edges we removed from G, this means that there exist three vertex sets V_1, V_2, V_3 such that this triangle has a vertex in each set and so that each of the bipartite graphs $(V_1, V_2), (V_2, V_3), (V_1, V_3)$ is $\varepsilon/4$-regular and has density at least $\varepsilon/2$. Lemma 6.6 now tells us that the number of triangles spanned by V_1, V_2, V_3 is at least

$$\frac{1}{2}\left(\frac{\varepsilon}{4}\right)^3 \left(\frac{n}{T\left(\frac{\varepsilon}{4}\right)}\right)^3 .$$

Since each of these triangles is also a triangle of G, this proves the lemma with $\delta(\varepsilon) = \frac{\varepsilon^3}{128T^3\left(\frac{\varepsilon}{4}\right)}$. □

We will now use the triangle-removal lemma to prove the following famous theorem.

Theorem 6.7 (Roth's Theorem)**.** *If $S \subseteq [n]$ does not contain 3-term arithmetic progression then $|S| = o(n)$. [Roth, 1953]*

Proof (by Ruzsa-Szemerédi): Given S we construct the following graph: G has $6n$ vertices in three sets A, B, C, where $|A| = n$, $|B| = 2n$ and $|C| = 3n$. We treat the vertices as integers. For every $s \in S$ and every $x \in [n]$ we put in G a triangle, denoted $T_{x,s}$ on the vertices $x \in A, x+s \in B, x+2s \in C$. For ease of reference, let us label the 3 edges of each of these $n|S|$ triangles $T_{x,s}$ with the integer s used to define them. We claim that the $n|S| = \varepsilon n^2$ triangles we put into G are edge disjoint. To see this, note that given any edge of the graph, we can uniquely recover the integers x and s that were used when placing it as part of the triangle $T_{x,s}$, implying that it belongs to a unique triangle. As the graph has εn^2 edge-disjoint triangles, it is clear that at least this many edges need to be removed in order to make it triangle free. By the Triangle Removal Lemma, we get that G has at least $c(\varepsilon)n^3$ triangles. For n sufficiently large $c(\varepsilon)n^3 > \varepsilon n^2$ hence we get a "new" triangle which is not one of those we explicitly constructed. In other words, we get a triangle, so that the labels on its three edges are s_1, s_2, s_3 and these three edges are not identical (if they were this would have been one of the triangles $T_{x,s}$ we placed in G). Suppose x is the vertex of this triangle in A. Then walking along the edges of this triangle, we see that $x + s_1 + s_2 - 2s_3 = x$, implying that $s_1 + s_2 = 2s_3$, that is, s_1, s_3, s_2 form a (non-trivial) 3-term arithmetic progression. □

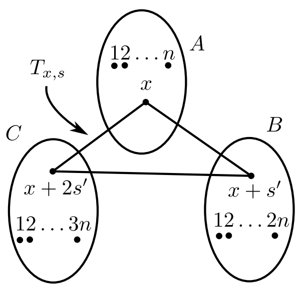


Figure 6.1: The graph with the $T_{x,s}$ triangle.

Let $\mathrm{twr}(x)$ be a tower of exponents of height x (so $\mathrm{twr}(4) = 2^{2^{2^2}}$). As we will see later on, the function $T(\varepsilon)$ from the statement of the Regularity Lemma is given by $\mathrm{twr}(1/\varepsilon^5)$. It follows that the function $\delta(\varepsilon)$ from the triangle-removal lemma is also a tower-type function. This in turn, implies that the above bound for Roth's Theorem gives that if $S \subseteq [n]$ does not contain a 3-term arithmetic progression then

$|S| = O\left(n/\log^* n\right)$, which is just "barely" $o(n)$. The best known upper bound for Roth's Theorem is $O\left(n(\log\log n)^5/\log n\right)$ and the best lower bound is $n/2^{c\sqrt{\log n}}$ (we will prove the lower bound later on).

A surprising result of Gowers showed that $T(\varepsilon) \geq \mathrm{twr}(\varepsilon^{-1/16})$, that is, that there are graphs with the property that every ε-regular graphs must be of size at least $\geq \mathrm{twr}(\varepsilon^{-1/16})$. The best known lower bound for the Triangle Removal Lemma is roughly $\delta(\varepsilon) > 1/\mathrm{twr}(\log 1/\varepsilon)$. Although this is still a tower-type bound, it is better than the $1/\mathrm{twr}(1/\varepsilon)$ bound that follows from the regularity lemma (as we have done above). The best upper bound for the triangle removal lemma is $(1/\varepsilon)^{\log(1/\varepsilon)}$ which is just barely super-polynomial. We will prove this bound in a later lecture.

6.3 Proof of the Regularity Lemma

Definition 6.8. Assume $P = \{V_1, V_2, \ldots, V_k\}$ is a partition of G to sets of size $a_1 n, \ldots, a_k n$, where $0 \leq a_i \leq 1$ and $\sum a_i = 1$. Define

$$q(V_i, V_j) = a_i a_j d^2(V_i, V_j)$$
$$q(P) = \sum_{i,j} q(V_i, V_j)$$

Note that since $0 \leq d(V_i, V_j) \leq 1$ we always have $0 \leq q(P) \leq 1$. The key part in proving the Regularity Lemma is the following lemma.

Lemma 6.9 (Key Lemma)**.** *If an equipartition $P = \{V_1, \ldots, V_k\}$ is not ε-regular and $k \geq \frac{1}{\varepsilon^6}$ then it is possible to form a new equipartition P''', such that $q(P''') \geq q(P) + \frac{\varepsilon^5}{2}$ and P''' has order at most $k^2 4^k$.*

Let us quickly derive the regularity lemma from the above lemma.

Proof (Regularity Lemma): We start with an arbitrary equipartition of order $\frac{1}{\varepsilon^6}$ and then repeatedly apply Lemma 6.9 until we get an ε-regular partition. As we observed above, we always have $0 \leq q(P) \leq 1$, and since each application of Lemma 6.9 increases $q(P)$ by at least $\varepsilon^5/2$ this process must stop after at most $\frac{2}{\varepsilon^5}$ iterations with a partition of order at most $\mathrm{twr}(4/\varepsilon^5)$. □

We turn to prove Lemma 6.9. We start with the following simple calculation.

Claim 6.10. *Assume A, B are vertex sets of sizes an, bn in some graph G, and assume A was partitioned into l subsets $A_1, \ldots, A_\ell$ of sizes $x_1 an, \ldots, x_\ell an$ and B was partitioned into ℓ subsets $B_1, \ldots, B_\ell$ of sizes $y_1 bn, \ldots, y_\ell bn$ (where $0 \leq x_i, y_i \leq 1$ and $\sum x_i = \sum y_i = 1$). Suppose for every i, j we have $d(A_i, B_j) = d(A, B) + \varepsilon_{ij}$ (ε_{ij} can be either positive or negative). Then we have*

$$\sum_{i,j} q(A_i, B_j) = q(A, B) + ab \sum_{i,j} x_i y_j \varepsilon_{ij}^2$$

Proof: Set $d = d(A, B)$ and just note that

$$\begin{aligned}\sum_{i,j} q(A_i, B_j) &= \sum_{i,j} x_i a y_j b \cdot d^2(A_i, B_j) = ab \sum_{i,j} x_i y_j (d + \varepsilon_{ij})^2 \\ &= ab \sum_{i,j} x_i y_j (d^2 + 2d\varepsilon_{ij} + \varepsilon_{ij}^2) = q(A, B) + 0 + ab \sum_{i,j} x_i y_j \varepsilon_{ij}^2\end{aligned}$$

where in the last equality we use the facts that $\sum_{i,j} x_i y_j = 1$, and that $\sum_{i,j} x_i y_j \varepsilon_{ij} = 0$. This last equality follows from the fact that

$$d = \frac{e(A,B)}{abn^2} = \frac{\sum_{i,j} e(A_i, B_j)}{abn^2} = \sum_{i,j} x_i y_j d(A_i, B_j) = \sum_{i,j} x_i y_j (d + \varepsilon_{ij}) = d + \sum_{i,j} x_i y_j \varepsilon_{ij} \ .$$

$\square$

The claim above gives us the following immediate corollaries:

Corollary 6.11. *If we take two vertex sets, A, B, participating in some partition P, and refine them into finer sets $A_1, \ldots, A_\ell$, and $B_1, \ldots, B_\ell$ then we have*

$$\sum_{i,j} q(A_i, B_j) \geq q(A, B) \tag{6.1}$$

Corollary 6.12. *Assume A, B are two vertex sets, participating in some equipartition P, and (A, B) is not ε-regular, namely there exists $A_1 \subseteq A$ and $B_1 \subseteq B$ such as $|A_1| \geq \varepsilon|A|$ and $|B_1| \geq \varepsilon|B|$ such that $|d(A_1, B_1) - d(A, B)| > \varepsilon$. Set $A_2 = A \setminus A_1$ and $B_2 = B \setminus B_1$. Then if P has order k (so $|A| = |B| = \frac{n}{k}$) then we have*

$$\sum_{1 \leq i,j \leq 2} q(A_i, B_j) \geq q(A, B) + \frac{\varepsilon^4}{k^2}$$

Now we can tackle the proof of Lemma 6.9.

Proof: Assume $P = \{V_1, \ldots, V_k\}$ is not ε-regular and denote by I the set of pairs (i, j) such that (V_i, V_j) is not ε-regular. So $|I| \geq \varepsilon k^2$. We construct a partition (not equipartition) P' satisfying $q(P') \geq q(P) + \varepsilon^5$ as follows. Assume (V_i, V_j) is not ε-regular. Then there are $V_1^{i,j} \subseteq V_i$ and $V_1^{j,i} \subseteq V_j$ which witness this fact, that is, they satisfy $|V_1^{i,j}| \geq \varepsilon|V_i|$, $|V_1^{j,i}| \geq \varepsilon|V_j|$ and $|d(V_i, V_j) - d(V_1^{i,j}, V_1^{j,i})| > \varepsilon$. Let us also set $V_2^{i,j} = V_i \setminus V_1^{i,j}$ and $V_2^{j,i} = V_j \setminus V_1^{j,i}$. Define P' to be the "atoms" of the Venn diagram of the sets $\{V_1, ..., V_k\}$ and the sets $\{V_1^{i,j}, V_1^{j,i}, V_2^{i,j}, V_2^{j,i} : (i, j) \in I\}$. Let k' denote the order of P'. Since for each set V_i we have at most $k - 1$ sets $V_1^{i,j}$, we get that each V_i is refined into at most 2^{k-1} sets, and so $k' \leq k2^{k-1}$.

We now compare $q(P')$ to $q(P)$. To this end, suppose each set V_i of P is refined in P' into sets $V_i^1, ..., V_i^\ell$. Note that for any j, the sets $V_i^1, ..., V_i^l$ refine the pair $V_1^{i,j}, V_2^{i,j}$. Corollary 6.11 tells us that for any i, j we have $\sum_{i',j'} q(V_i^{i'}, V_j^{j'}) \geq q(V_i, V_j)$. Furthermore, if $(i, j) \in I$ then applying Corollary 6.12 and then Corollary 6.11 we get

$$\sum_{i',j'} q(V_i^{i'}, V_j^{j'}) \geq q(V_1^{i,j}, V_1^{j,i}) + q(V_1^{i,j}, V_2^{j,i}) + q(V_2^{i,j}, V_1^{j,i}) + q(V_2^{i,j}, V_2^{j,i}) \geq q(V_i, V_j) + \varepsilon^4/k^2 \ .$$

In other words, we recover all the contributions $q(V_i, V_j)$ and when (i, j) is not ε-regular we in fact gain at least ε^4/k^2. Since we assume that there are at least εk^2 pairs that are not ε-regular we conclude that $q(P') \geq q(P) + \varepsilon^5$.

We are now just left with turning P' into an equipartition while not decreasing q by much. We start with partitioning each cluster of P' into clusters of size $n/(k')^2$. After sequentially "pulling" from each of the clusters of $V_i' \in P'$ subsets of size $n/(k')^2$, we will eventually be left with a remainder set U_i of size less than $n/(k')^2$. Let P'' be the new partition[10]. As P'' is a refinement of P' we get from Corollary 6.11 that we still have $q(P'') \geq q(P') \geq q(P) + \varepsilon^5$.

[10]Note that apart from the sets $U_1, \ldots, U_{k'}$ the partition P'' is an equipartition.

Assume $P'' = \{V'_1, \ldots, V'_{k''}, U_1, \ldots, U_{k'}\}$ where $V_1, \ldots, V_{k''}$ are the subsets of size exactly $\frac{n}{(k')^2}$ and $U_1, \ldots, U_{k'}$ are the remainders (of size $< \frac{n}{(k')^2}$). Note that we also have $\sum_i |U_i| \le k' \cdot \frac{n}{(k')^2} = \frac{n}{k'} \le \varepsilon^6 n$. To get a true equipartition we now simply "distribute" the vertices of $U_1 \cup \ldots \cup U_{k'}$ equally among the sets $V_1, ..., V_{k''}$ thus obtaining an equipartition P''' of order at most $(k')^2 \le k^2 4^k$. It should be clear that since $\sum_i |U_i| \le \varepsilon^6 n$ this cannot decrease $q(P'')$ by much. In fact, it follows from Exercise 6.13 below that $q(P''') \ge q(P'') - 8\varepsilon^6$. Since we can always assume that ε is small enough (make sure you see why), we get

$$q(P''') \ge q(P'') - 8\varepsilon^6 \ge q(P) + \varepsilon^5 - 8\varepsilon^6 \ge q(P) + \varepsilon^5/2 \,,$$

as needed. □

Exercise 6.13. Suppose $P = \{V_1, \ldots, V_k, U_1, \ldots, U_t\}$ is a partition of $V(G)$ with $|V_1| = \cdots = |V_k|$ and $\sum_{i=1}^{t} |U_i| \le \varepsilon n$. Show how to turn P into an equipartition P of order k satisfying $q(P') \ge q(P) - 8\varepsilon$.

7 Seventh Lecture

7.1 The Counting Lemma

The following lemma, which is usually referred to as the counting lemma, states that if a collection of vertex sets are regular/dense enough then we can find in them any small subgraphs we would expect to find, if the graphs were genuinely random. It generalizes a similar lemma we proved last time for triangles. A naive generalization would state the following.

Theorem 7.1. *For every d and h there is are $\varepsilon = \varepsilon(d, h)$ and $c = c(d, h)$, so that if all pairs in $V_1, \ldots, V_h$ are of size at least c, of density at least d and are ε-regular, then $V_1, \ldots, V_h$ contain a copy of any graph H on h vertices (with one vertex in each cluster V_i).*

Exercise 7.2. Prove the above assertion.

The shortcoming of the above version is that we (unnecessarily) insist on using one vertex from each cluster. Also, note that ε depends on h. We will instead prove the following more refined version.

Theorem 7.3 (Embedding Lemma)**.** *For every $d > 0$ and Δ there exists $\varepsilon = \varepsilon(d, \Delta)$ and $c = c(d, \Delta)$ so that the following holds: Assume $V_1, \ldots, V_r$ are vertex-sets of size at least ch such that for every i, j the pair (V_i, V_j) is ε-regular and satisfies $d(V_i, V_j) \ge d$. Then, if H is a graph on at most h vertices with $\chi(H) \le r$ and $\Delta(H) \le \Delta$ then $V_1, \ldots, V_r$ span a copy of H.*

We will prove that there are in fact $c(\varepsilon, h)|V|^h$ copies of H (assuming all sets are of the same size $|V|$). We will first need the following simple observation regarding ε-regular pairs.

Claim 7.4. *Assume (A, B) is an ε-regular pair with density d. Then, if $B' \subseteq B$ is of size at least $\varepsilon|B|$, then there are at most $\varepsilon|A|$ vertices in A that have less than $(d - \varepsilon)|B'|$ neighbors in B'.*

Proof: Otherwise, taking A' to be the subset of vertices with less than $(d - \varepsilon)|B'|$ neighbors in B' we see that A', B' contradict the ε-regularity of (A, B). □

Proof (Embedding Lemma): Assume an r-coloring of H's vertices induces the partition into r vertex-disjoint sets $U_1, \ldots, U_r$. We shall find an embedding of H in $V_1, ..., V_h$, where the vertices of U_i are chosen out of V_i. Namely, every vertex $v_i \in H$ has a set $V_{\sigma(i)}$ in which it shall be embedded. The idea of the proof is to simply embed the vertices $v_1, \ldots, v_h$ in a "greedy" way. We will just have to show that we never get "stuck".

Suppose that for some $0 \leq j \leq h-1$ we have already found vertices $x_1, ..., x_j$ in $V_{\sigma(1)}, ..., V_{\sigma(j)}$ so that x_i, x_j are connected if v_i, v_j are connected in H. For every $i > j$ we define a subset $Y_i^j \subseteq V_{\sigma(i)}$ of all the possible vertices which can be chosen for v_i given the choice of $x_1, ..., x_j$, namely all the vertices x that satisfy the following condition: if $j' \leq j$ and v_i is connected to $v_{j'}$ in H then x is connected to $x_{j'}$. Observe that for $j = 0$ we can set $Y_i^0 = V_{\sigma(i)}$ for every $1 \leq i \leq h$.

For every $0 \leq j < i \leq h$ let $b_{j,i}$ denote the number of vertices among $v_1, ..., v_j$ that are adjacent to v_i in H. We claim that if we choose ε appropriately then we can pick the vertices $x_1, ..., x_h$ so that for every $0 \leq j < i \leq h$ we have $|Y_i^j| \geq (d-\varepsilon)^{b_{j,i}} |V_{\sigma(i)}|$. This clearly holds for $j = 0$ (since $b_{0,i} = 0$) so assuming it holds for $j-1$ we prove it for j. In this case we need to pick a vertex $x_j \in Y_j^{j-1} \subseteq V_{\sigma(j)}$ that will play the role of v_j. The definition of Y_j^{j-1} guarantees that any vertex in this set is adjacent to the appropriate vertices among $x_1, ...x_{j-1}$. Once we pick a vertex x_j we will need to set Y_i^j (with $i > j$) to be the subset of vertices of Y_i^{j-1} that are adjacent to x_j. Recall that we know from induction that $|Y_i^{j-1}| \geq (d-\varepsilon)^{b_{j-1,i}} |V_{\sigma(i)}|$ for every $i \geq j$. This means that if $i > j$ and v_i is not adjacent to v_j then after picking x_j we can take $Y_i^j = Y_i^{j-1}$ so the condition still holds for such vertices. As to i for which v_i is adjacent to v_j, note that in this case we just need to make sure that $|Y_i^j| \geq (d-\varepsilon)|Y_i^{j-1}|$. Observe that if $\varepsilon \leq (d/2)^\Delta$ then since $b_{i,j} \leq \Delta$ we get from induction that

$$|Y_i^{j-1}| \geq (d-\varepsilon)^{b_{j-1,i}} |V_{\sigma(i)}| \geq (d-\varepsilon)^\Delta |V_{\sigma(i)}| \geq \varepsilon |V_{\sigma(i)}| \,.$$

We thus get from Claim 7.4, with A, B being the sets $V_{\sigma(j)}$,$V_{\sigma(i)}$ and $B' = Y_i^{j-1}$, that $V_{\sigma(j)}$ contains at most $\varepsilon|V_{\sigma(j)}|$ vertices that have less than $(d-\varepsilon)|Y_i^{j-1}|$ neighbors in Y_i^{j-1}. Since H has maximum degree Δ we get that all but $\Delta\varepsilon|V_{\sigma(j)}|$ of the vertices in $V_{\sigma(j)}$ have at least $(d-\varepsilon)|Y_i^{j-1}|$ neighbors in each of the sets Y_i^{j-1} for which v_i is a neighbor of v_j in H. Note that choosing any of these vertices will make sure that $|Y_i^j| \geq (d-\varepsilon)|Y_i^{j-1}|$ as we wanted. We thus get that we have at least

$$|Y_j^{j-1}| - \Delta\varepsilon \left|V_{\sigma(j)}\right| \geq (d-\varepsilon)^{b_{j-1,j}} |V_{\sigma(j)}| - \Delta\varepsilon \left|V_{\sigma(j)}\right| \geq (d-\varepsilon)^\Delta \left|V_{\sigma(j)}\right| - \Delta\varepsilon \left|V_{\sigma(j)}\right| \geq \varepsilon |V_{\sigma(j)}| \,,$$

choices for a vertex $x_j \in Y_j^{j-1}$ satisfying the required condition. The only thing we need to make sure is that ε is small enough to that the last inequality holds. It is easy to check that taking $\varepsilon = (d/2)^\Delta / 2\Delta$ satisfies this inequality as well as the previous one we wanted ε to satisfy. Finally, note that up to $h-1$ of the vertices of Y_{j-1}^j might have already been chosen as vertices $x_{j'}$ for some $j' < j$ so we need to make sure that $\varepsilon|V_{\sigma(j)}| - h \geq 1$. Taking $c = 1/\varepsilon$ makes sure this condition holds. This completes the proof of the induction.

We finally note that if each set is of size at least $(2/\varepsilon)h$ then $\varepsilon|V_{\sigma(j)}| - h \geq (\varepsilon/2)|V_{\sigma(j)}|$ so at each iteration we actually have at least $(\varepsilon/2)|V_{\sigma(j)}|$ choices for the vertex x_j. □

7.2 Another Proof of the Erdős-Stone-Simonovits Theorem

The embedding lemma gives us a pretty simple proof of the Erdős-Stone-Simonovits:

Theorem. *For every b, r and $\delta > 0$ there exists $n_0(\delta, r, b)$ so that if G is a graph on at least n_0 vertices with $\left(1 - \frac{1}{r} + \delta\right) \frac{n^2}{2}$ edges then G contains a copy of K_{r+1}^b (a b-blowup of K_{r+1}).*

Proof: Define $\varepsilon = \varepsilon(\frac{\delta}{8}, rb)$ and $c = c(\frac{\delta}{8}, rb)$ and apply the Regularity Lemma with $\min(\varepsilon, \frac{\delta}{8})$. We a get an ε-regular partition into k sets $V_1, \ldots, V_k$ where $\frac{1}{\varepsilon} \leq k \leq T(\varepsilon)$ and therefore $|V_i| \geq \frac{n}{T(\varepsilon)}$. Remove the following edges from the graph:

1. Edges within V_i. There are at most $\frac{\delta n^2}{8}$ such edges.

2. Edges connecting V_i to V_j where $d(V_i, V_j) < \frac{\delta}{8}$. There are at most $\frac{\delta n^2}{4}$ such edges.

3. Edges connecting V_i to V_j where (V_i, V_j) is not ε-regular. There are at most $\frac{\delta n^2}{8}$ such edges.

All in all, we removed less than $\frac{\delta n^2}{2}$ edges. Therefore G still has at least $\left(1 - \frac{1}{r}\right) \frac{n^2}{2}$ edges, so by Turán's Theorem (weak version) G contains a copy of K_{r+1}. This K_{r+1} must have a single vertex in each one of $r+1$ sets $V_1, \ldots, V_{r+1}$ so that all pairs (V_i, V_j) are ε-regular and satisfy $d(V_i, V_j) \geq \frac{\delta}{8}$. We have $r+1$ sets of size at least $\frac{n}{T(\varepsilon)}$ such as the density between them is at least $\frac{\delta}{8}$ and every one of them is $\varepsilon(\frac{\delta}{8}, rb)$-regular. The graph K^b_{r+1} is an $(r+1)$-colorable graph with maximal degree rb. Hence, the Embedding Lemma would imply that $V_1, \ldots, V_{r+1}$ have a copy of K^b_{r+1} if

$$|V_i| \geq \frac{n}{T(\varepsilon)} \geq c\,(\delta/8, rb) \cdot (r+1)b$$

So to conclude the proof, we just set

$$n_0 = c\,(\delta/8, rb) \cdot b(r+1) \cdot T\left(\min\left\{\delta/8, \varepsilon\,(\delta/8, rb)\right\}\right) .$$

Since n_0 depends only on δ, r and b the proof is complete. $\square$

Recall that the Triangle Removal Lemma, states that for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such as if one needs to remove from G at least εn^2 edges in order to make it triangle free, then G has at least δn^3 triangles. This can be extended to any fixed H as follows.

Theorem (Graph Removal Lemma)**.** *For every $\varepsilon > 0$ and graph H, there exists $\delta = \delta_H(\varepsilon)$ such that if one needs to remove from G at least εn^2 edges in order to make it H-free, then G has at least δn^h copies of H (where $h = |V(H)|$).*

Exercise 7.5. Prove the above assertion.

7.3 Ramsey Numbers of Bounded Degree Graphs

We can state Ramsey's Theorem as follows.

Theorem (Ramsey)**.** *Every 2-coloring of K_{4^n} has a monochromatic copy of K_n.*

One can thus consider the more general function $R(K, H)$ which is the smallest integer N so that every 2-coloring of K_N contains either a red K or a black H. Of course, if both K and H are of size n then we have the trivial bound $R(K, H) \leq R(K_n, K_n) \leq 4^n$ but it is reasonable to suspect that for "simple" graphs one should be able to come up with better bounds. As it turns out, if the graphs have bounded degree, then it is enough to consider a 2-coloring of the complete graph on $O(n)$ vertices (as opposed to the 4^n bound for K_n).

Theorem 7.6 (Chvátal, Rödl, Szemerédi, Trotter '83)**.** *For every Δ there is $c' = c'(\Delta)$ so that if K, H are graphs on n vertices with maximal degree at most Δ then $R(H, K) \leq c'n$.*

Proof: Set $\varepsilon = \varepsilon(\frac{1}{2}, \Delta)$, $c = c(\frac{1}{2}, \Delta)$, and $m = 4^{\Delta+1}$. Given a 2-coloring of $K_{c'n}$ we apply the regularity lemma with $\varepsilon' = \min\left\{\varepsilon, \frac{1}{4m}\right\}$ on the black edges. Observe that since we are considering a coloring of the complete graph, then if the black edges form an ε'-regular bipartite graph between (V_i, V_j) then so do the red edges. We get a regular partition of size at least $T(\varepsilon') \geq 2m$. There are $k \geq 2m$ sets and at most $\frac{k^2}{4m}$ of the pairs are not regular. Therefore at least $\frac{k^2}{2}(1 - \frac{1}{2m})$ pairs are regular. Define a graph on k vertices and connect vertex i to j iff (V_i, V_j) is an ε'-regular pair. This graph has at least $\binom{k}{2} - \frac{k^2}{4m}$ edges. Since $k \geq 2m$ we have $\binom{k}{2} - \frac{k^2}{4m} \geq \frac{k^2}{2}\left(1 - \frac{1}{m}\right)$, so by Turán's Theorem this graph contains K_m, meaning that there exists m subsets $V_1, \ldots, V_m$ for which every pair is ε'-regular.

Consider the following 2-coloring of a graph on $m = 4^{\Delta+1}$ vertices; (i,j) is colored green if $d(V_i, V_j) \geq \frac{1}{2}$ and (i,j) is blue if $d(V_i, V_j) < \frac{1}{2}$. By Ramsey's Theorem, this coloring contains a monochromatic $K_{\Delta+1}$. This gives us $\Delta + 1$ sets $V_1, \ldots, V_{\Delta+1}$ so that every pair is ε'-regular and one of the following holds:

1. All the densities of black edges are at least $\frac{1}{2}$.

2. All the densities of red edges are at least $\frac{1}{2}$.

Assuming case 1 above holds (case 2 is identical), we now apply the Embedding Lemma on $V_1, \ldots, V_{\Delta+1}$ and find a black copy of H (case 2 is identical, in which case we find a red copy of K). Since H has maximal degree Δ it is $\Delta + 1$ colorable. The choice of ε also guarantees that the pairs are regular and dense enough for an application of the Embedding Lemma. We only need to make sure the sets V_i are large enough, that is that

$$|V_i| \geq \frac{|G|}{T(\varepsilon')} \geq c \cdot n \,.$$

Hence, we can set $c'(\Delta) = c \cdot T(\varepsilon')$. □

8 Eighth Lecture

We continue with more applications of the Regularity Lemma.

8.1 The Induced Ramsey Theorem

Recall the Regularity Lemma and the Embedding Lemma:

Theorem. *For every $\varepsilon > 0$ exists $T = T(\varepsilon)$ such as every graph has an ε-regular equipartition of size k where $\frac{1}{\varepsilon} \leq k \leq T(\varepsilon)$.*

Theorem. *For every $d > 0$ and Δ exists $\varepsilon = \varepsilon(d, \Delta)$ and $c = c(d, \Delta)$ such that if $V_1, \ldots, V_r$ are vertex-sets of size $\geq ch$ such as $d(V_i, V_j) \geq d$ for every i, j and also every pair (V_i, V_j) is ε-regular, then for every graph H, where $|H| \leq h$ vertices with $\chi(H) \leq r$ and $\Delta(H) \leq \Delta$, we can find an embedding of H in $V_1, \ldots, V_r$.*

We now state an induced version of the Embedding Lemma:

Theorem 8.1 (Induced Embedding Lemma)**.** *For every h there exists constants $\varepsilon' = \varepsilon'(h)$ and $c' = c'(h)$ such that if $V_1, \ldots, V_h$ are vertex sets of size $\geq c'$ and every pair (V_i, V_j) is ε'-regular with density $\in \left[\frac{1}{8}, \frac{7}{8}\right]$ then it is possible to embed in $V_1, \ldots, V_h$ an* induced *copy of H for every graph $|H| \leq h$.*

Exercise 8.2. Prove the above Induced Embedding Lemma. The case of $H = K_h$ follows as a special case of the Embedding Lemma we have previously proved. The case of general H can be reduced to the case of K_h.

We recall the statement of Ramsey's Theorem.

Theorem (Ramsey)**.** *For every t, there exists $N = N(t)$ so that every 2-coloring of K_N has a monochromatic copy of K_t.*

It is natural to look for an induced version of Ramsey's Theorem. That is, given a fixed graph H we want to find a graph $R = R(H)$ so that in any 2-coloring of $E(H)$ one can find a monochromatic induced copy of H. Note that Ramsey's Theorem states that we can take K_{4^n} as $R(K_n)$. Observe that for any other H, the graph $R(H)$ cannot be a complete graph. It was an open problem for some time whether $R(H)$ indeed exists for every H. The following theorem states that they indeed exist.

Theorem 8.3 (Induced Ramsey Theorem)**.** *For every graph H there exists a graph $R = R(H)$ such that in every 2-coloring of $E(R)$ it is possible to find a monochromatic induced copy of H.*

Proof: The idea is very simple; we will show that a large enough random graph can be taken as the graph R. Of course, we need to define some deterministic property. So set $\varepsilon = \varepsilon'(h)$, $c = c'(h)$, $m = 4^h$, and $T = T\left(\min\left\{\frac{\varepsilon}{2}, \frac{1}{4m}\right\}\right)$. We will show that if G is a graph on n vertices such that:

1. $n/T \geq c$
2. For every pair (A, B) of disjoint sets of size $\geq \frac{\varepsilon n}{T}$ it holds that $\left|d(A, B) - \frac{1}{2}\right| \leq \varepsilon/2$

then G can be taken as the graph $R(H)$. Note that we claim that a single graph can be used for all H on h vertices. Observe that what the second condition guarantees is that *every* pair of sets A, B of size at least n/T would be an ε-regular pair of density $1/2$.

Let us first explain why there are graphs satisfying the above two conditions. The first condition just asks the graph to be large enough. For the second condition we note that by a simple averaging argument, if the condition $\left|d(A, B) - \frac{1}{2}\right| \leq \varepsilon/2$ holds for subsets A, B of size k then it also holds for bigger subsets as well. Now, a simple applications of Chernoff's inequality gives that in $G(n, 1/2)$, with high probability every pair of sets A, B of size $100 \log n/\varepsilon^2$ satisfy $\left|d(A, B) - \frac{1}{2}\right| \leq \varepsilon/2$. So we just need to take n to be large enough so that the first condition will hold, and also large enough so that $\varepsilon n/T \geq 100 \log n/\varepsilon^2$, thus making sure that the second condition also holds. So we see that indeed almost all (large enough) graphs satisfy the above two conditions.

We now turn to show that if R is a graph satisfying the above two conditions then it satisfies the assertions of the theorem. So consider a red/black coloring of $E(R)$, and apply the Regularity Lemma on the black edges with error parameter $\min\left\{\frac{\varepsilon}{2}, \frac{1}{4m}\right\}$. The lemma will return an equipartition of the vertices to k sets where $2m \leq k \leq T$ so that at most $\frac{k^2}{4m}$ of the pairs are not $\frac{\varepsilon}{2}$-regular. A simple, yet important observation is that if (V_i, V_j) is $\varepsilon/2$-regular with respect to the black edges, then our assumption on G guarantees that (V_i, V_j) is also ε-regular with respect to the red edges (verify this!).

By Turán's Theorem (weak version) we get m sets $V_1, \ldots, V_m$ such that every pair (V_i, V_j) is ε-regular (in both colors). Now observe that our second assumption about G implies that all pairs satisfy $d(V_i, V_j) \in [0.49, 0.51]$ (assuming $\varepsilon \leq 1/50$ which we can clearly impose). Ramsey's Theorem then implies that those $4^h = m$ sets contain h sets $V_1, \ldots, V_h$ where every pair (V_i, V_j) is ε-regular as above but also one of the following holds:

(a) In all the pairs, the density of the black edges is $\in [1/4, 0.51]$.

(b) In all the pairs, the density of the black edges is $\in [0, 1/4]$.

Assume case (a) holds, and consider the vertex sets $V_1, \ldots, V_h$. We now pick some of the edges between these clusters as follows. For every (i, j):

1. If $(i, j) \in H$ then take the black edges between V_j and V_i.
2. If $(i, j) \notin H$ then take the black and red edges between V_j and V_i.

We claim that the Induced Embedding Lemma can now be used in order to find an induced black copy of H. This is obvious for $(i,j) \in H$ since we assume that in these cases the black edges in (V_i, V_j) have density $\in [0.25, 0.51]$ and that they form and ε-regular bipartite graph. For $(i,j) \notin H$ it follows from condition 2 on the graph G that when taking all (i.e. black and red) edges between (V_i, V_j) that the resulting graph is ε-regular, since all the sets in the partition are of size at least n/T. As to the density, we know from condition 2 that $d(V_i, V_j) \in [0.49, 0.51]$. Finally, note that the first condition on G gives that $|V_i| \geq \frac{n}{T} \geq c$, so we see that we can indeed apply the Induced Embedding Lemma to get an induced black copy of H in G.

Assume now that case (b) holds. As before consider the vertex sets $V_1, \ldots, V_h$, and recall that in this case all pairs (V_i, V_j) are ε-regular with respect to the red edges. Pick some of the edges between these clusters as follows. For every (i,j):

1. If $(i,j) \in H$ then take the red edges between V_j and V_i.
2. If $(i,j) \notin H$ then take the black and red edges between V_j and V_i.

We claim that the Induced Embedding Lemma can now be used in order to find an induced red copy of H. This is obvious for $(i,j) \in H$ since we assume that in these cases the red edges in (V_i, V_j) have density $\in [0.24, 0.51]$ and that they form and ε-regular bipartite graph. For $(i,j) \notin H$ it follows from condition 2 on the graph G that when taking all (i.e. black and red) edges between (V_i, V_j) that the resulting graph is ε-regular, since all the sets in the partition are of size at least n/T. As to the density, we know from condition 2 that $d(V_i, V_j) \in [0.49, 0.51]$. Finally, note that the first condition on G gives that $|V_i| \geq \frac{n}{T} \geq c$, so we see that we can indeed apply Induced Embedding Lemma to get an induced red copy of H in G. $\square$

8.2 The $(6,3)$-Problem and the Induced Matchings Problem

Definition 8.4 ((6,3) Problem)**.** Let $f(n)$ denote the largest number of edges a 3-uniform hypergraph on n vertices can have, if it does not contain 6 vertices that span at least 3 edges.

Definition 8.5 (Induced Matching Problem)**.** Let $g(n)$ be the maximal number of edges in a graph on n vertices whose edges can be partitions into n induced matchings.

Definition 8.6 (One Edge One Triangle Problem)**.** Denote by $h(n)$ the maximal number of edges in a graph on n vertices where each edge belongs to exactly one triangle.

Theorem 8.7. *We have the following equivalence between the above three problem $h = \Theta(g) = \Theta(f)$.*

We will apply the following simple but very useful claim.

Claim 8.8. *Every graph has a bipartite subgraph with $\frac{m}{2}$ edges.*

Proof: Partition $V(G)$ randomly to two sets A, B. For every edge (xy), the probability x, y are in different partitions is $\frac{1}{2}$. Hence $\mathbb{E}[m] = \frac{m}{2}$, therefore a partition with at least this many edges exists. $\square$

Exercise. Find an explicit way (i.e. an efficient deterministic algorithm) to produce a partition of $V(G)$ so that at least half the edges cross the cut.

We will prove the equivalence between h, g, f in the following three claims.

Claim. $g(n) \leq 2f(2n)$.

Proof: Assume G is a graph on n vertices and $M_1, \ldots, M_n$ is a partition of its vertices into induced matchings. We take a bipartite subgraph of G with at least half of the edges. We now construct a hypergraph H on $2n$ vertices whose number of edges will be the same as the number of edges in G and will fulfill the $(6,3)$-Problem. Such a construction will prove that $\frac{g(n)}{2} \leq f(2n)$. Let $A = \{v_1, \ldots, v_n\}$ be a set of n new vertices. For every i, and for every edge $(x, y) \in M_i$ we put in H the edge (v_i, x, y).

We now claim that H does not contain 6 vertices spanning 3 edges. To see this, take any 6-tuple of vertices and consider the following possibilities:

1. Since a vertex in A forms a 3-edge with a matching of edges in G, it cannot be the case that only one of the vertices is from A.
2. If 2 vertices are in A then we have 2 vertices on each side of G. If 2 of the edges belong to the same matching, then we cannot have a third on the 4 vertices of G by the induced matching property. To have 3 edges from 3 different matching would require 3 distinct vertices from A which is a contradiction.
3. It is easy to see that we cannot have at most 3 vertices in $V(G)$.

Thus, no set of 6 vertices in H spans 3 edges, giving $\frac{g(n)}{2} \leq f(2n)$. □

Claim. $h(n) \leq g(n)$

Proof: Assume G fulfills the condition that every edge belongs to one triangle. Define n matchings $M_1, \ldots, M_n$ in the following way: For every i M_i would contain the edges spanned by $N(v_i)$. Note that

$$\left| \bigcup M_i \right| = |E(G)|$$

as every edge belongs to one triangle, and thus is part of the neighborhood of the opposed vertex. It is easy to verify that every M_i is indeed an induced matching. Therefore, $h(n) \leq g(n)$. □

Claim. $f(n) \leq h(n) + n$

Proof: Assume H is a 3-uniform hypergraph which fulfills the $(6,3)$-condition. As long as there is a vertex whose degree is 1, remove the vertex and the relevant edge. Assume we performed k such iterations. The new 3-graph, H', fulfills the $(6,3)$ condition as a subgraph of H. Note that if $m = |E(H)|$ then $|E(H')| = m - k$ and that the $(6,3)$ property of H' and the fact that each vertex of H' has min-degree 2, implies that $(*) : H'$ does not contain 2 edges sharing 2 vertices.

Define a graph G on n vertices in the following way: For every edge in H' we put in G a triangle on the same vertices. Property $(*)$ implies that $|E(G)| = 3|E(H')| = 3(m-k)$. Property $(*)$ also implies that every edge of G belongs to precisely one triangle. This means that $3(m-k) \leq h(n-k) \leq h(n)$ implying that $m \leq h(n) + n$. □

Claim. *The Triangle Removal Lemma implies that* $h(n) = o(n^2)$.

Proof: Assume G fulfills the OEOT condition and has εn^2 edges. Then G is the disjoint union of $\frac{\varepsilon n^2}{3}$ edge disjoint triangles, implying that G is $\frac{\varepsilon}{3}$-far from being triangle-free. Therefore, by the Triangle Removal Lemma G has at least $\delta(\varepsilon)n^3$ triangles. For sufficiently large n, we have $\delta(\varepsilon)n^3 \geq \frac{\varepsilon n^2}{3}$. Hence, there exists an "extra" triangle. This triangle must have edges from at least two different triangles (in fact, three), making those edges belong to two different triangles, which is a contradiction. □

The result of $f(n) = o(n^2)$ was first proved by [Ruzsa and Szemerédi, 1978]. Recall that by the equivalences we proved above, we also get that $g(n) = o(n^2)$ and $f(n) = o(n^2)$.

9 Ninth Lecture

9.1 Behrend's Construction

In the last lecture we considered three extremal problems; The $(6,3)$ Problem, The Induced Matching Problem and The One Edge One Triangle Problem. Denoting by $f(n), g(n), h(n)$ their respective extremal functions, we proved that $f = \Theta(g) = \Theta(h) = o(n^2)$. What the proof actually gave, is that $h(n) = O\left(n^2/\left(\log^* n\right)^{1/5}\right)$, which is just barely sub-quadratic. We will first prove a lower bound for these three extremal-problems. We first prove a very surprising relation between them and the question of bounding $r_3(n)$.

Theorem 9.1. *Let $r_3(n)$ denote the size of the largest subset of $\{1,\ldots,n\}$ that contains no 3-term arithmetic progression. Then $h(n) \geq n \cdot r_3(n)$.*

We will then prove the following lower bound on $r_3(n)$, due to Behrend.

Theorem 9.2 ([Behrend, 1946])**.** $r_3(n) \geq n/2^{c\sqrt{\log n}} = n^{1-o(1)}$.

As an immediate corollary of the above two theorems, we will get the following lower bounds for f, g, h.

Corollary 9.3. $f(n), g(n), h(n) \geq n^2/2^{c\sqrt{\log n}} = n^{2-o(1)}$.

Corollary. *Combining the results we proved we get the following diagram of implications*

$$\begin{array}{c} TRL \rightarrow OEOT \rightarrow r_3(n) = o(n) \\ \downarrow \searrow \\ IM \leftarrow (6,3) - P \end{array}$$

Proof (Theorem 9.1): We prove that $h(6n) \geq 3nr_3(n)$, using the same idea we used in the proof of Roth's Theorem. Assume $S \subseteq \{1,\ldots,n\}$ is of size $r_3(n)$ and does not contain 3-term arithmetic progressions. We construct a graph on $6n$ vertices as follows: For every $1 \leq x \leq n$ and every $s \in S$ we add a triangle on the vertices $x \in A, x+s \in B, x+2s \in C$ (see Figure 6.1) and label its edges by s. As in the proof of Roth's Theorem, it is easy to see that these triangles are edge-disjoint, and that if the edges of some triangle are labeled by s_1, s_2, s_3 then $s_1 + s_2 = 2s_3$. As S does not contain 3-term arithmetic progressions, it follows that necessarily $s_1 = s_2 = s_3$ and thus any triangle is one of the original triangles we put in the graph. These two observations imply that each edge belongs to exactly one triangle. $\square$

Corollary 9.4. $(6,3)$ *Problem $\rightarrow$ Roth's Theorem ($r_3(n) = o(n)$).*

Proof (Theorem 9.2): Denote $n = d^k$. We shall think about $x \in \{1,\ldots,n\}$ in base d, hence it has k digits. Define

$$\bar{S}_{d,k,r} = \left\{ (x_1,\ldots,x_k) \mid \left(0 \leq x_i < \frac{d}{2}\right) \wedge \sum_{i=1}^{k} x_i^2 = r^2 \right\}$$

$$S_{d,k,r} = \left\{ \sum_{i=1}^{k} x_i d^{k-1} \mid (x_1,\ldots,x_k) \in \bar{S}_{d,k,r} \right\}$$

Note that

$$\left|\bigcup_{r\geq 0} \bar{S}_{d,k,r}\right| = (d/2)^k = n/2^k \ ,$$

since the union of these sets has all vectors x with digits $0 \leq x_i < \frac{d}{2}$. Since $|\bar{S}_{d,k,r}| = |S_{d,k,r}|$ we get that the total size of the sets $S_{d,k,r}$ is also $n/2^k$. We will shortly show that each set $S_{d,k,r}$ does not contain a 3-term arithmetic progression, but let us first show how to finish the proof given this fact. To this end we just need to pick d, k, r. For every d, k the value of $r^2 = \sum_{i=1}^{k} x_i^2$ is an integer in $[0, kd^2]$. Since the total size of the sets $S_{d,k,r}$ is $n/2^k$ so for some r we have $|S_{d,k,r}| \geq n/k2^k d^2$. If we pick $k = \sqrt{\log n}$ and $d = 2^{\sqrt{\log n}}$ we get that at least one of the sets $S_{d,k,r}$ has no 3-term arithmetic progression and is of the required size.

We now prove the required claim. Assume $x, y, z \in S_{d,k,r}$ and $x + y = 2z$. By rewriting the numbers in base d we get that $\bar{x} + \bar{y} = 2\bar{z}$ that is, that $x_i + y_i = 2z_i$ for every i (this follows from the fact that all the digits belong to $[0, \frac{d}{2}]$ hence there is no carry). $\bar{S}_{d,k,r}$ is actually a sphere, hence no three distinct points can be on the same line, hence $\bar{x} = \bar{y} = \bar{z}$ and from here we get $x = y = z$. More detailed version: $x_i + y_i = 2z_i$, hence by Jensen's inequality $x_i^2 + y_i^2 \geq 2z_i^2$ and summing over all digits we get $\sum x_i^2 + \sum y_i^2 \geq 2\sum z_i^2$. But since these three terms should sum to r^2, we must have $x_i^2 + y_i^2 = 2z_i^2$ for every i which together with $x_i + y_i = 2z_i$ implies that $x_i = y_i = z_i$. $\square$

Fact 9.5. *The best known bounds for $r_3(n)$ are $n/2^{c\sqrt{\log n}} \leq r_3(n) \leq O\left(n(\log\log n)^5/\log n\right)$.*

9.2 Lower Bound for the Triangle Removal Lemma

The lower-bound for $r_3(n)$ gives us a lower-bound for the $(6,3)$-Problem. Now we shall seek a lower bound for the Triangle Removal Lemma.

Theorem 9.6. *For every $\varepsilon > 0$ exists $\delta = \delta(\varepsilon) > 0$ such as if G is a graph which at least εn^2 edges must be removed before it becomes triangle free, then G has at least δn^3 triangles.*

Unfortunately, as we mentioned in an earlier lecture, the best known lower bound are of the form $\delta(\varepsilon) \geq 1/\mathrm{twr}(1/\varepsilon)$. It is thus natural to ask if a more civilized bound can be obtained. For example, is it true that $\delta(\varepsilon) \geq \varepsilon^C$ for some absolute constant C? The following theorem gives a negative answer.

Theorem 9.7. *For all small enough $\varepsilon > 0$ we have $\delta(\varepsilon) \leq \varepsilon^{c\log\left(\frac{1}{\varepsilon}\right)}$. Namely, there exist graphs for which εn^2 edges need to be removed to make them triangle-free but they contain "only" $\varepsilon^{c\log\left(\frac{1}{\varepsilon}\right)} n^3$* triangles.

Proof: We take a graph H on m vertices from the lower bound of OEOT (see theorem 9.1), so it has $\frac{m^2}{2^{c\sqrt{\log m}}}$ triangles, where each edge is part of exactly one triangle. Now we take $\frac{n}{m}$-blowup of H to the graph G. We note:

1. G has n vertices.
2. G has $\left(\frac{n}{m}\right)^3 \frac{m^2}{2^{c\sqrt{\log m}}} = \frac{n^3}{m2^{2\sqrt{\log m}}}$ triangles.
3. In order to remove all the triangles in a blowup of an original triangle we need to remove at least $\frac{n^2}{m^2}$ edges (all the edges between two sets of size $\frac{n}{m}$), hence over all we need to remove at least $\frac{n^2}{m^2}\frac{m^2}{2^{c\sqrt{\log m}}} = \frac{n^2}{2^{c\sqrt{\log m}}}$ edges (as the original triangles are edge-disjoint so we need to remove the blowups of all the original triangles).

So we need to remove $\frac{n^2}{2^{c\sqrt{\log m}}}$ edges in order to make G triangle-free and G has $\leq \frac{n^3}{m}$ triangles. Hence, in order to make sure G is ε-far from being triangle free we pick m to be the largest integer satisfying $2^{c\sqrt{\log m}} \geq \frac{1}{\varepsilon}$. It is easy to see that $m \geq \left(\frac{1}{\varepsilon}\right)^{c\log\left(\frac{1}{\varepsilon}\right)}$, which completes the proof. □

Alon proved that for a fixed graph H, the removal lemma for H has a poly(ε) bound if and only if H is bipartite.

9.3 Deducing Szemerédi's Theorem from the Hypergraph Removal Lemma

We have seen that the Regularity Lemma implies the Triangle Removal Lemma, which in turn implies Roth's Theorem. A famous theorem of Szemerédi extends Roth's theorem by showing that for any fixed k, if $S \subseteq [n]$ does not contain a k-term arithmetic progression then $|S| = o(n)$. It is thus natural to ask if one can come up with a graph theoretic proof of Szemerédi's Theorem using the Regularity Lemma. As it turns out, this seems to be impossible. Therefore we shall attempt to do so using hypergraphs. To this end one first needs to prove a removal lemma for hypergraphs. Let us state such a removal lemma for k-uniform hypergraphs and for the graph K^k_{k+1} that is the complete k-uniform hypergraph on $k+1$ vertices.

Theorem 9.8 (Hypergraph Removal Lemma)**.** *If one should removal at least εn^k edges from a k-uniform hypergraph H in order to make it K^k_{k+1}-free then H contains at least $\delta(\varepsilon, k) \cdot n^{k+1}$ copies of K^k_{k+1}.*

Our goal now is to prove the following:

Theorem 9.9. *The Hypergraph removal lemma implies Szemerédi's Theorem.*

Before we prove the theorem, let us comment on the proof of the Hypergraph Removal Lemma. Recall that our plan for proving the Triangle Removal Lemma consisted of the following steps:

1. We defined the concept of "graph regularity".
2. We proved the Regularity Lemma, which states that every graph has an equipartition satisfying the notion of "graph regularity" we came up with.
3. We proved a counting lemma, that is, that 3 vertex sets that satisfy our notion of "graph regularity" contain many copies of K_3.

When one tries to extend this plan and prove the Hypergraph Removal Lemma, one first needs to come up with a concept of hypergraph regularity. The first choice that comes to mind is the following:

Definition 9.10 (Regularity condition for 3-uniform hypergraph)**.** A, B, C are ε-regular if for every $A' \subseteq A, B' \subseteq B, C' \subseteq C$ such that $|A'| \geq \varepsilon|A|$, $|B'| \geq \varepsilon|B|$, $|C'| \geq \varepsilon|C|$ we have

$$|d(A', B', C') - d(A, B, C)| \leq \varepsilon$$

where $d(A, B, C) = \frac{e(A,B,C)}{|A||B||C|}$.

It is easy to adapt the proof of the graph regularity lemma and prove the following.

Theorem 9.11. *Every 3-uniform hypergraph has an ε-regular partition of size at most $T\left(\frac{1}{\varepsilon^5}\right)$.*

The problem with this regularity lemma is that the concept of regularity it uses is not strong enough to satisfy the third condition of our plan. That is, it is not true that if 4 vertex sets A, B, C, D are such that each 3 of them are dense and regular then they contain many copies of K^3_4. In fact, they might not contain any copy of K^3_4! Here is how to construct such an example. Take 4 vertex sets of size n and for every pair of vertices (in two different vertex sets) pick a direction randomly, uniformly and independently. We take x, y, z to be an edge iff they form a directed cycle. Hence x, y, z is an edge with probability $\frac{1}{4}$ and it is easy to verify that each of the 4 triples of vertex sets are $o(1)$-regular (as the density in every triplet of vertex-sets of size $\Omega(n)$ will be $\frac{1}{4} \pm o(1)$ with high probability). But it is also easy to see that this 3-uniform hypergraph has no copy of K_4^3.

One can also come up with very strong notions of regularity that satisfy condition 3, but do not satisfy condition 2, that is, that it is not the case that all hypergraphs have an equipartition satisfying these conditions. We finally note that it is possible to define a "right" notion of regularity and thus prove the removal lemma, but it is rather complicated and hence we shall omit it.

We now return to our original task of deducing Szemerédi's Theorem from the hypergraph removal lemma. We will do so for 4-term arithmetic progressions. The proof for general k is identical and is thus left as an exercise.

Proof (Theorem 9.9): Given $S \subset \{1, \ldots, n\}$ we define a 3-uniform hypergraph H over 4 vertex-sets X_1, X_2, X_3, X_4, each of size $O(n)$. For every $x_1, x_2 \in [n]$ and $s \in S$ we place a copy of K_4^3 whose vertices are $x_1 \in X_1$, $x_2 \in X_2$, $a_1x_1 + a_2x_2 + s \in X_3$, and $b_1x_1 + b_2x_2 + s \in X_4$, where a_1, a_2, b_1, b_2 are absolute constants (i.e. independent of n and S) that will be determined later. For ease of reference, we label the 4 edges of this copy with the integer s. Note that we have placed in H exactly $n^2|S|$ copies of K_4^3 (one for every $x_1, x_2 \in [n]$ and $s \in S$). We will shortly prove that:

Claim 9.12. *If $a_1 \neq b_1$ and $a_2 \neq b_2$ then these $|S|n^2$ copies of K_4^3 are edge disjoint.*

Since $|S| \geq \varepsilon n$ this implies that one should remove from H at least εn^3 edges in order to make it K_4^3-free, implying that H is ε/c-far from being K_4^3-free (the c comes from the fact that H has $O(n)$ vertices). Hence by the Hypergraph Removal Lemma we get that H contains $\delta(\varepsilon)n^4$ copies of K_4^3.

The key part of the proof will be the following:

Claim 9.13. *It is possible to choose $a_1 \neq b_1$ and $a_2 \neq b_2$ so that if we take the labels s_1, s_2, s_3, s_4 written on the 4 edges of some copy of K_4^3 in H, then s_1, s_2, s_3, s_4 form a 4-term arithmetic progression.*

Therefore, as soon as n is large enough so that $\delta(\varepsilon)n^4 \geq \varepsilon n^3 \geq n^2|S|$ we get a copy of K_4^3 whose edges are not marked by the same s, which gives us a non-trivial 4-term arithmetic progression. So given the above claims the proof is complete. □

We now finish the proof of Theorem 9.9 by proving the two claims we stated above.

Proof (Claim 9.12): It is sufficient to show that given an edge it is possible to determine x_1, x_2, s. If the edge contains a vertex from X_1 and X_2 this is obvious. If the edge contains vertices $x_1 \in X_1$, $x_3 \in X_3$ and $x_4 \in X_4$ we can recover x_2 and s by solving the linear equations (observe that we "know" x_1, x_3 and x_4)

$$\begin{cases} a_1x_1 + a_2x_2 + s & = x_3 \\ b_1x_1 + b_2x_2 + s & = x_4 \end{cases}$$

Therefore if $a_2 \neq b_2$ it is possible to uniquely determine x_1, x_2 and s. If the edge has vertices in X_2, X_3 and X_4 then we need the condition $a_1 \neq b_1$. So the copies of K_4^3 are edge-disjoint if $a_1 \neq b_1$ and $a_2 \neq b_2$. □

Proof (Claim 9.13): Note that $z_1 \leq z_2 \leq z_3 \leq z_4$ form a 4-term arithmetic progression iff they satisfy the linear equations $z_1 + z_3 = 2z_2$ and $z_2 + z_4 = 2z_3$. By performing elementary row operations we get that $z_1 \leq z_2 \leq z_3 \leq z_4$ form a 4-term arithmetic progression iff they satisfy the linear equations

$$z_1 - 2z_2 = -z_3 \quad \text{and} \quad 2z_1 - 3z_2 = -z_4$$

Assume $x_1 \in X_1, x_2 \in X_2, x_3 \in X_3, x_4 \in X_4$ span a copy of K_4^3. Suppose the edge (x_1, x_2, x_3) is marked by s_1 and the edge (x_1, x_2, x_4) is marked by s_2. Then we know that

$$a_1x_1 + a_2x_2 + s_1 = x_3 \quad \text{and} \quad b_1x_1 + b_2x_2 + s_2 = x_4$$

Suppose the edge (x_1, x_3, x_4) is marked by s_3. This edge was originally defined using x_1, s_3 and some integer x_2'. Hence,

$$a_1x_1 + a_2x_2' + s_3 = x_3 \quad \text{and} \quad b_1x_1 + b_2x_2' + s_3 = x_4$$

Combining these two ways of expressing x_3 and x_4 we deduce that

$$a_2(x_2 - x_2') = s_3 - s_1 \quad \text{and} \quad b_2(x_2 - x_2') = s_3 - s_2$$

Multiplying the first equation by $-b_2$, the second by a_2 and then summing them we see that s_1, s_2, s_3 satisfy the linear equation $b_2s_1 - a_2s_2 = (b_2 - a_2)s_3$. Recall that our goal is to show that s_1, s_2, s_3 satisfy the linear equation $z_1 - 2z_2 = -z_3$ so setting $b_2 = 1$ and $a_2 = 2$ we get the required equation. Note that this choice satisfies the requirement of the previous claim that $b_2 \neq a_2$. By an identical argument, this time using the edge (x_2, x_3, x_4) instead of the edge (x_1, x_3, x_4), we get that $b_1s_1 - a_1s_2 = (b_1 - a_1)s_4$. Our goal is to show that s_1, s_2, s_4 satisfy the linear equation $2z_1 - 3z_2 = -z_3$ so setting $b_1 = 2$ and $a_1 = 3$ we get the required equation (note that again $a_1 \neq b_1$). Since s_1, s_2, s_3, s_4 satisfy the two linear equations that define a 4-term arithmetic progression, the proof is complete. □

10 Tenth Lecture

10.1 The Ramsey-Turán Problem

We know that a graph can be triangle-free and have $n^2/4$ edges, but such a graph contains large independent sets. We thus ask the following: can a graph G be triangle-free, have only independent sets of size $o(n)$ and still have cn^2 edges, or does G necessarily have $o(n^2)$ edges? It is easy to see that the answer is the latter. Indeed, if $\alpha(G) \leq \delta n$ then $\Delta(G) \leq \delta n$ (as an edge in the neighborhood of a vertex implies a triangle hence all the neighborhood of each vertex must be independent set). So we get that $2m \leq n\delta n$, namely if G is K_3-free and $\alpha(G) = o(n)$ then $m = o(n^2)$.

But what if instead of K_3-free graphs we asked about K_5-free graphs? The answer is given by the next two theorems:

Theorem 10.1. *There exists a K_5-free graph, that contains only independent sets of size $o(n)$ and contains more than $\frac{n^2}{4}$ edges.*

Theorem 10.2. *Every graph with $\geq \left(\frac{1}{4} + \varepsilon\right) n^2$ edges contains either a K_5 or independent set of size $\delta(\varepsilon) \cdot n$.*

Note that Turán's Theorem guarantees a copy of K_3 when $m > \frac{1}{4}n^2$. The above theorem states that if we forbid independent sets of size $\Omega(n)$ then $m > (\frac{1}{4} + \varepsilon)n^2$ edges give us a K_5.

Proof (Theorem 10.1): We start with $K_{n/2,n/2}$ with bipartition into two sets A, B. This already gives us $\frac{n^2}{4}$ edges. We now wish to find a graph H on $n/2$ vertices that will have the following two properties

1. All independent sets in H will be of size $o(n)$.

2. H will be triangle-free.

It is clear that if we put one such H in A and one in B, the final graph will satisfy all the required properties.

It is not hard to show that such an H exists. In fact it is known that there exists a K_3-free graph where all independent sets are of size $O(\sqrt{n \log n})$. Since we do not need such a strong result (which is rather hard to prove), we will be content with showing that there is a K_3-free graph where all independent sets are of size $n^{2/3}$. To this end we take $G(n,p)$. Since we want the graph to be triangle-free we would like to have

$$\mathbb{E}[\#K_3] = p^3 n^3 < \frac{n}{4} ,$$

so we set $p = \frac{1}{2n^{2/3}}$. The probability to have an independent set of size n^c is at most

$$\binom{n}{n^c}\left(1 - \frac{1}{2n^{2/3}}\right)^{n^{2c}} \leq 2^{n^c \log n} e^{-\frac{n^{2c}}{n^{2/3}}} ,$$

so we would like that $n^{2/3} n^c \leq n^{2c}$. Hence picking any $c > 2/3$, gives that with high probability the graph has no independent set of size n^c and that with probability at least $1/2$ will have at most $n/2$ copies of K_3 (we use Markov's Inequality here). So with positive probability we get a graph with no independent set of size n^c and with at most $n/2$ copies of K_3. Removing one vertex from each of these K_3's gives the required graph. $\square$

Proof (Theorem 10.2): Given G with $\geq \left(\frac{1}{4} + \varepsilon\right) n^2$ edges we need to find in it either a K_5 or an independent set of size $\delta(\varepsilon)n$. We now apply Claim 2.3, which tells us that G contains a subgraph G' on $n' \geq \varepsilon n$ vertices with $\delta(G') \geq \left(\frac{1}{2} + \frac{\varepsilon}{2}\right) n'$. So it is enough to find an independent set of size $\delta(\varepsilon)n'$ in G' (or a K_5). For the sake of simplicity we assume that G itself satisfies the condition $\delta(G) \geq \left(\frac{1}{2} + \varepsilon\right) n$. For every edge (x,y), the vertices x,y have at least εn common neighbors. If those vertices do not contain any edge then we get an independent set of size εn, and we are done. So suppose they contain an edge z,w, which gives us a copy of K_4 on the vertices x,y,z,w. As $\delta(G) \geq \left(\frac{1}{2} + \varepsilon\right) n$ the number of edges between x,y,z,w and the rest of the vertices is at least $4\left(\frac{n}{2} + n\varepsilon\right) = 2n + 4\varepsilon n$. If there is a vertex connected to all the 4 vertices x,y,z,w, it is a copy of K_5 and we are done. Otherwise, denote by U those vertices connected to 3 of the vertices x,y,z,w. The number of edges e between $\{x,y,z,w\}$ to the rest of the vertices thus satisfies

$$2n + 4\varepsilon n \leq e \leq 3|U| + 2(n - |U|) = |U| + 2n$$

implying that $|U| \geq 4\varepsilon n$. At least $1/4$ of the vertices in U must be connected to the same triplet of vertices from x,y,z,w. This set of vertices, call it U', has size at least εn. Finally, if U' contains an edge we get a K_5, and if it has no edge we get an independent set of size εn. $\square$

Having addressed the problem for K_3 and K_5 we now address the case of K_4. It turns out that answering such questions for even K_r's is more difficult than for odd K_r's. For K_4 the answer turned out to be around $n^2/8$ edges.

Theorem 10.3. *If G contains $\left(\frac{1}{8} + \varepsilon\right) n^2$ edges then G either contains K_4 or an independent set of size $\delta(\varepsilon)n$ (this time $\delta(\varepsilon)$ is something like $1/twr(1/\varepsilon)$).*

Theorem 10.4. *For every $\varepsilon > 0$, and large enough $n > n_0(\varepsilon)$, there exists a K_4-*free *graph with $\left(\frac{1}{8} - \varepsilon\right) n^2$ edges that contains only independent set of size $o(n)$.*

We will not prove the above lower bound, and instead address Theorem 10.3. We will need a few preliminary claims.

Claim 10.5. *Assume H is a graph on two vertex-sets A, B of size m each, and $d(A,B) \geq \frac{1}{2} + \varepsilon$. Then H contains either K_4 or an independent set of size $\geq \varepsilon m$.*

Proof: It is easy to see that A has at least $\frac{\varepsilon m}{2}$ vertices with at least $\left(\frac{1}{2} + \frac{\varepsilon}{2}\right) m$ neighbors in B. Indeed,

$$\frac{\varepsilon m}{2} \cdot m + \left(1 - \frac{\varepsilon}{2}\right) m \left(\frac{1}{2} + \frac{\varepsilon}{2}\right) m < m^2 \left(\frac{1}{2} + \varepsilon\right) .$$

If those high degree vertices do not span an edge then we're done (we found an independent set of size $\frac{\varepsilon m}{2}$). Otherwise, assume x, y are two such vertices, and they are connected. Then they have $\frac{\varepsilon m}{2}$ common neighbors in B. Then those vertices are either an independent set of size $\frac{\varepsilon m}{2}$ (and we are done) or they contain an edge, which gives a copy K_4 together with x, y. $\square$

Claim 10.6. *If A, B, C are vertex-sets of size m such as (A,B), (A,C) and (B,C) are ε-regular with densities $\geq 3\varepsilon$, then they contain either a K_4 or an independent set of size $\varepsilon^2 m$.*

Proof: Since (A,B) is ε-regular with density 3ε all the vertices of A, apart of εm of them, have a minimal degree $\geq 2\varepsilon m$ in B. The same goes for C. In particular, there exists a vertex a (actually almost all of them) whose degree in B and C is at least $2\varepsilon m$. Denote the neighbors of a in B and C by B' and C'. Since $|C'| \geq \varepsilon |C|$, all of B's vertices, apart for possibly εm of them, have minimal degree $(3\varepsilon - \varepsilon)|C'| \geq \varepsilon^2 |C|$ in C'. Since $|B'| \geq 2\varepsilon |B|$, we infer that B' has a vertex b whose degree in C' is at least $\varepsilon^2 |C|$. We call those neighbors C''. We know that a is connected to b, and both of them are connected to every vertex in C''. Hence, we either have an independent set of size $\varepsilon^2 m$ on C'' or a copy of K_4. $\square$

Claim 10.7. *If G has $\left(\frac{1}{8} + \varepsilon\right) n^2$ edges then it either contains K_4 or independent set of size δn where $\delta = \frac{\varepsilon^2}{T(\varepsilon/6)}$.*

Proof: Given G we apply the Regularity Lemma with $\varepsilon/6$. We remove from G the following edges:

1. Edges between non-$\varepsilon/6$-regular pairs ($\leq \varepsilon n^2/6$ edges).
2. Edges within the sets of the partition ($\leq \varepsilon n^2/12$ edges).
3. Edges between pairs with density $\leq \frac{\varepsilon}{2}$ ($\leq \varepsilon n^2/4$ edges).

All in all, we removed at most $\frac{\varepsilon n^2}{2}$ edges. The remaining graph, G', thus has at least $\left(\frac{1}{8} + \frac{\varepsilon}{2}\right) n^2$ edges. All the pairs in the equipartition must have density $\leq \left(\frac{1}{2} + \frac{\varepsilon}{2}\right)$ else by Claim 10.5 we would infer that G contains K_4 or an independent set of size $\geq \frac{\varepsilon}{2} \frac{n}{T}$.

How many pairs (V_i, V_j) with density > 0 exist in G'? If the partition has k sets then no more than $\frac{k^2}{4}$, otherwise Mantel's Theorem guarantees 3 sets such that the density between any two of them is $\geq \varepsilon/2$ and they are $\varepsilon/6$-regular. But given such a triple, Claim 10.6 would guarantee existence of either K_4 or an independent set of size $\frac{\varepsilon^2}{9} \frac{n}{T}$ in G. So the number of edges in G' is at most

$$\frac{k^2}{4} \left[\frac{n^2}{k^2} \left(\frac{1}{2} + \frac{\varepsilon}{2}\right)\right] = \frac{n^2}{8} + \frac{\varepsilon n^2}{8} < \left(\frac{1}{8} + \frac{\varepsilon}{2}\right) n^2 ,$$

contradicting the fact that $e(G') > \left(\frac{1}{8} + \frac{\varepsilon}{2}\right) n^2$. Hence, somewhere along the way we aught to have found either a K_4 or an independent set of size $\delta(\varepsilon) \cdot n$. $\square$

10.2 Quasi-Random Graphs

Consider the following proposed definition:

Definition. We shall say a graph is (ε, p)-quasi-random if every set S contains $\binom{|S|}{2}(p \pm \varepsilon)$ edges (where $x \pm \varepsilon$ is a number between $x - \varepsilon$ and $x + \varepsilon$).

The above definition is flawed, as by Ramsey's Theorem it will not hold for sets of size $o(\log n)$. So we consider the following three alternative definitions:

Definition 10.8. We say G fulfills P_1 if every set S contains $p\binom{|S|}{2} \pm \varepsilon n^2 = p\frac{|S|^2}{2} \pm \varepsilon n^2$ edges.

Definition 10.9. We say G fulfills P_2 if every set S of size $\geq \varepsilon n$ contains $\frac{|S|^2}{2}(p \pm \varepsilon)$ edges.

Definition 10.10. We say G fulfills P_3 if it fulfills P_2 for sets of size $\frac{n}{2}$.

We say that $P_1 \implies P_2$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if G fulfills P_1 with an error of δ then it fulfills P_2 with an error of ε. We now wish to prove the following equivalence.

Theorem 10.11. $P_1 \iff P_2 \iff P_3$

We break the proof into several claims.

Claim 10.12. $P_1 \implies P_2$

Proof: We note that P_2 is fulfilled iff it holds for sets of size εn (by averaging argument it will hold for any bigger set). In this case we require that

$$e(S) = p\frac{|S|^2}{2} \pm \varepsilon|S|^2 = p\frac{|S|^2}{2} \pm \varepsilon^3 n^2$$

so it is enough that P_1 will hold with $\delta = \varepsilon^3$. $\square$

Claim 10.13. $P_2 \implies P_1$

Proof: If G fulfills P_2 with δ then P_1 holds for every S of size $\geq \delta n$ since $\frac{\delta|S|^2}{2} \leq \delta n^2$. So if $\delta \leq \varepsilon$ then P_1 holds for all subsets of size at least δn. For sets of size at most δn we would like that P_1 will be fulfilled trivially. Namely, that

$$\frac{p|S|^2}{2} - \varepsilon n^2 \leq 0 \leq e(S) \leq \frac{|S|^2}{2} \leq \frac{p|S|^2}{2} + \varepsilon n^2$$

So to get the left hand side we need to have $\frac{p|S|^2}{2} - \varepsilon n^2 \leq 0$ so we also ask than $|S| \leq \delta n \leq \sqrt{\frac{2\varepsilon}{p}}n$, that is, that $\delta \leq \sqrt{\frac{2\varepsilon}{p}}$. For the other side we would need to have $\frac{p|S|^2}{2} + \varepsilon n^2 \geq \frac{|S|^2}{2}$ so we also ask that $|S| \leq \delta n \leq \sqrt{\frac{2\varepsilon}{1-p}}n$, that is, that $\delta \leq \sqrt{\frac{2\varepsilon}{1-p}}$. So overall it is enough to take $\delta = \min\left(\varepsilon, \sqrt{\frac{2\varepsilon}{p}}, \sqrt{\frac{2\varepsilon}{1-p}}\right)$. $\square$

Claim 10.14. $P_2 \iff P_3$

Proof: It is obvious that $P_2 \implies P_3$. We will prove the other direction by showing that if G does not fulfill P_2 with error ε then it does not fulfill P_3 with some other small error $\delta(\varepsilon)$. Recall that if G fails to satisfy P_2 with error ε then there exists a set A of size εn whose density is not $p \pm \varepsilon$. We assume that it is $= p + \varepsilon$ (the case $p - \varepsilon$ is similar). We will show that in this case G has a subset of size $n/2$ with density $p + \Omega(\varepsilon^4)$. This means that if G satisfies P_3 with error $\delta(\varepsilon) = \varepsilon^4$ then it must satisfy P_2 with error ε.

Denote $B = V \backslash A$. If the density of B is not $(p \pm \delta)$ then we are done as by averaging, it must contain a subset of size $\frac{n}{2}$ whose density is also not in $(p \pm \delta)$. Since G has $pn^2/2$ edges we get that the density of the bipartite graph connecting A and B is given by

$$d(A,B) = \frac{\frac{pn^2}{2} - e(A) - e(B)}{\varepsilon n(1-\varepsilon)n} = \frac{\frac{pn^2}{2} - \frac{\varepsilon^2 n^2}{2}(p+\varepsilon) - \frac{(1-\varepsilon)^2 n^2}{2}(p \pm \delta)}{\varepsilon n(1-\varepsilon)n}$$

We pick a random subset $S \subseteq B$ of size $(1/2 - \varepsilon)n$ and we examine the expected number of edges contained in $A \cup S$. This is a set of size $\frac{n}{2}$ hence it is sufficient to show that the expected value is $\geq (p + \Omega(\varepsilon^3))\frac{n^2}{8} > (p+\delta)\frac{n^2}{8}$ which will imply that P_3 is not fulfilled with error $\delta = \varepsilon^4$. The expected number of edges in A is just the number of edges in A which is $\frac{\varepsilon^2 n^2 p}{2} + \frac{\varepsilon^3 n^2}{2}$. The expected number of edges in S is

$$\mathbb{E}[e(S)] = \frac{(1/2-\varepsilon)^2 n^2}{2}(p \pm \delta) ,$$

and the expected number of edges between A and S is

$$\mathbb{E}[e(A,S)] = \varepsilon n(\frac{1}{2} - \varepsilon)n \frac{\left[\frac{pn^2}{2} - \frac{\varepsilon^2 n^2}{2}(p+\varepsilon) - \frac{(1-\varepsilon)^2 n^2}{2}(p \pm \delta)\right]}{\varepsilon n(1-\varepsilon)n} .$$

So the total expected number of edges in $A \cup S$ is

$$\frac{\varepsilon^2 n^2 p}{2} + \frac{\varepsilon^3 n^2}{2} + \frac{(1/2-\varepsilon)^2 n^2}{2}(p \pm \delta) + \varepsilon n(\frac{1}{2} - \varepsilon)n \frac{\left[\frac{pn^2}{2} - \frac{\varepsilon^2 n^2}{2}(p+\varepsilon) - \frac{(1-\varepsilon)^2 n^2}{2}(p \pm \delta)\right]}{\varepsilon n(1-\varepsilon)n} .$$

We want to show that this expression is at least $(p + \Omega(\varepsilon^3))\frac{n^2}{8}$, so dividing both sides by $n^2/2$ we need to show that

$$\varepsilon^2 p + \varepsilon^3 + (1/2-\varepsilon)^2(p \pm \delta) + \frac{(\frac{1}{2} - \varepsilon)}{(1-\varepsilon)}\left[p - \varepsilon^2 p - \varepsilon^3 - (1-\varepsilon)^2(p \pm \delta)\right] \geq \frac{p}{4} + \Omega(\varepsilon^3) .$$

As we assumed $\delta = \varepsilon^4$ all the terms involving δ are $O(\varepsilon^4)$ so we can disregard them as all the rest are of order at least ε^3. So we are down to proving that

$$\varepsilon^2 p + \varepsilon^3 + (1/2-\varepsilon)^2 p + \frac{(\frac{1}{2} - \varepsilon)}{(1-\varepsilon)}\left[p - \varepsilon^2 p - \varepsilon^3 - (1-\varepsilon)^2 p\right] \geq \frac{p}{4} + \Omega(\varepsilon^3) .$$

The expression in the square brackets can now be further simplified to $p(2\varepsilon - 2\varepsilon^2) - \varepsilon^3$ implying that the entire LHS is just $p/4 + \varepsilon^3 - \varepsilon^3 \frac{(\frac{1}{2}-\varepsilon)}{(1-\varepsilon)} > p/4 + \varepsilon^3/2$, and the proof is complete. □

11 Eleventh Lecture

Throughout this lecture we use $d(x,y)$ to denote the co-degree of x, y, that is, the number of vertices that are connected to both x and y.

11.1 The Chung-Graham-Wilson Theorem

Definition 11.1. A graph G is said to satisfy property P_4 if it has $pn^2/2$ edges and the number of copies of C_4 (i.e. the 4-cycle) in G is at most $(p^4+\varepsilon)n^4$. Here, the number of C_4 is the number of ordered 4-tuples (x,y,z,w) so that $(x,z),(x,w),(y,z),(y,w)$ are all edges in the graph.

The advantage of this notion of quasi-randomness, compared to the previous properties we considered last lectures, is that it can easily verified in polynomial time, compared to properties P_1, P_2, P_3 which require checking exponentially many conditions.

Theorem 11.2 (Chung-Graham-Wilson Theorem)**.** $P_3 \iff P_4$ *[Chung et al., 1989]*

The implication $P_3 \implies P_4$ is the easy part of this theorem.

Exercise 11.3. Show that $P_3 \implies P_4$.

Exercise 11.4. Let P_3' be the property of having at most $(p^4+\varepsilon)n^4$ copies of C_4. Does $P_3' \implies P_4$?

We shall thus focus on proving that $P_4 \implies P_3$. As a warm-up, we start with the following claims.

Claim 11.5. *Every graph with $pn^2/2$ edges contains at least p^2n^3 copies of $K_{1,2}$, where a copy of $K_{1,2}$ is an ordered 3-tuple (x,y,z) such that both (x,z) and (y,z) are edges in the graph.*

Proof: $\#K_{1,2} = \sum d^2(x) \geq n\left[\frac{\sum d(x)}{n}\right]^2 = p^2n^3.$ □

Claim 11.6. *Every $n\times n$ bipartite graph on vertex sets A,B, that has pn^2 edges contains at least p^2n^3 copies of $K_{1,2}$, with one vertex in A and 2 in B .*

Proof: Identical to the previous proof. □

Claim 11.7. *If G contains K copies of $K_{1,2}$ then it contains at least $(K/n)^2$ copies of C_4.*

Proof: $\#C_4 = \sum_{x,y} d^2(x,y) \geq n^2\left[\frac{\sum d(x,y)}{n^2}\right]^2 = n^2\left(\frac{K}{n^2}\right)^2 = \left(\frac{K}{n}\right)^2.$ □

Claim 11.8. *Every graph with $pn^2/2$ edges contains at least $(pn)^4$ copies of C_4.*

Proof: Immediately from Claims 11.5 and 11.7. □

As a warm up towards proving Theorem 11.2, let us prove that P_4 forces the graph to be nearly regular.

Claim 11.9. *If G satisfies P_4 then all but δn vertices have degree $(p\pm\delta)n$.*

Proof: Assume the contrary and recall that if $\frac{1}{n}\sum_{i=1}^n x_i = d$, then setting $x_i = d+\varepsilon_i$ we have

$$\sum_{i=1}^n x_i^2 = d^2n + \sum_{i=1}^n \varepsilon_i^2 \; . \tag{11.1}$$

Hence, if we denote $d(x) = pn + \varepsilon_x n$ then we get

$$\#K_{1,2} = \sum_x d^2(x) = \sum_x (p+\varepsilon_x)^2n^2 = p^2n^3 + \sum_x \varepsilon^2 n^2 \geq p^2n^3 + \delta^3n^3 \; .$$

We can now infer from Claim 11.7 that

$$\#C_4 \geq \left(\frac{p^2n^3 + \delta^3 n^3}{n}\right)^2 \geq (p^4 + 2p^2\delta^3)n^4 \ ,$$

which is a contradiction to the P_4 property of G, upon taking $\varepsilon = 2p^2\delta^3$. $\square$

Our plan for proving Theorem 11.2 is similar to the one we employed in the last proof, that is, we show that if G fails to satisfy P_3 then it has more than $(pn)^4$ copies of C_4. Let us first observe that the above claim does not prove that $P_4 \implies P_3$. Indeed, while one expects a quasi-random graph to be nearly regular, it is easy to see that being regular is not equivalent to properties P_1, P_2, P_3. For example $K_{n/2,n/2}$ is $n/2$-regular but fails to satisfy these three properties. In other words, a graph can fail to satisfy P_3 but still have the correct number of $K_{1,2}$. In a more technical level, this means that we will have to rework the proof of Claim 11.7 rather than rework the proof of Claim 11.5 as we have done in the above claim.

Proof (Chung-Graham-Wilson Theorem): We show that if G fails to satisfy P_3 then it fails to satisfy P_4 as well. Assume G has $pn^2/2$ edges and contains a vertex set A of size $n/2$ whose density is not in $p \pm \varepsilon$. We need to show that G contains at least $(p^4 + \delta)n^4$ copies of C_4 for some $\delta = \delta(\varepsilon) > 0$. Set $B = V \setminus A$ and suppose $e(A) = an^2/8$, $e(B) = bn^2/8$ and $e(A, B) = cn^2/4$. Then we have $a + 2c + b = 4p$. Since we assume that $a \notin p \pm \varepsilon$ we get from (11.1) that

$$a^2 + 2c^2 + b^2 \geq 4p^2 + \varepsilon^2 \ . \tag{11.2}$$

Claim 11.7 tells us that G has at least p^2n^3 copies of $K_{1,2}$. Our goal is thus to improve the argument of Claim 11.7 by finding $\sim n^2$ pairs of vertices whose co-degree deviates from its expectation (which is p^2n) by $\sim n$.

To this end we will focus on the pairs of vertices x, y so that either $x, y \in A$ or $x, y \in B$. We first calculate $\sum_{x,y\in A} d(x, y)$, which equals the number of copies of $K_{1,2}$ (namely, triples (x, y, z) with both (x, z) and (y, z) edges of the graph), with $x, y \in A$. By Claim 11.7, applied to the graph induced by A, we get that the number of copies of $K_{1,2}$ with all three vertices in A is at least $a^2n^3/8$. By Claim 11.6, applied to the bipartite graph connecting A to B, we get that the number of copies of $K_{1,2}$ with two vertices in A and one in B is at least $c^2n^3/8$. So we get that $\sum_{x,y\in A} d(x, y) \geq (a^2 + c^2)n^3/8$. An identical argument implies that $\sum_{x,y\in B} d(x, y) \geq (b^2 + c^2)n^3/8$. These two facts, together with (11.2), imply that

$$\sum_{(x,y)\in A^2\cup B^2} d(x, y) \geq (a^2 + b^2 + 2c^2)n^3/8 \geq p^2n^3/2 + \varepsilon^2n^3/8. \tag{11.3}$$

Denote $d(x, y) = p^2n + \delta_{xy}$. Then (11.3) is equivalent to saying that

$$\sum_{(x,y)\in A^2\cup B^2} \delta(x, y) \geq (n^2/2) \cdot \varepsilon^2n/4. \tag{11.4}$$

Hence, we get

$$\begin{aligned}\#C_4 &= \sum_{x,y} d^2(x, y) = \sum_{x,y} \left(p^2n + \delta_{xy}\right)^2 = p^4n^4 + \sum_{x,y} \delta_{x,y}^2 \geq p^4n^4 + \sum_{(x,y)\in A^2\cup B^2} \delta_{xy}^2 \\ &\underset{\substack{\uparrow\\ \text{(Jensen)}}}{\geq} p^4n^4 + \frac{n^2}{2}\left(\frac{\sum \delta_{xy}}{n^2/2}\right)^2 \underset{\substack{\uparrow\\ (11.4)}}{\geq} p^4n^4 + \frac{n^2}{2}(\varepsilon^2n/4)^2 = p^4n^4 + \varepsilon^4n^4/32 \ .\end{aligned}$$

So we can finally conclude the proof by setting $\delta = \varepsilon^4/32$. $\square$

11.2 An Algorithmic Version of the Regularity Lemma

Inspecting the proof of the regularity lemma, one sees that the proof actually supplies an algorithm that given ε and a graph G, constructs an ε-regular partition of G. However, there is one part which is not "explicit" in that it is not clear how to implement it in polynomial time. The part we are referring to is where given a bipartite graph on vertex sets A, B that is not ε-regular, we need to find two vertex sets $A' \subseteq A$ and $B' \subseteq B$ of sizes $|A'| \geq \varepsilon|A|$ and $|B'| \geq \varepsilon|B|$ satisfying $d(A', B') \notin d(A, B) \pm \varepsilon$. It is easy to see that if we could produce these sets in polynomial time, then the running time of the entire process would also be polynomial[11]. As it turns out solving this algorithmic problem is hard. Instead we will show the following:

Theorem 11.10. *There is an $O(n^3)$ time algorithm that given a bipartite graph G on vertex sets A, B does the following; if G is not ε-regular, it finds two vertex sets $A' \subseteq A$ and $B' \subseteq B$ of sizes $|A'| \geq \varepsilon^5|A|$ and $|B'| \geq \varepsilon^5|B|$ satisfying $d(A', B') \notin d(A, B) \pm \varepsilon^5$.*

Note that the algorithm does not produce a pair of sets A', B' satisfying the properties we used in the proof of the regularity lemma, since the ε is replaced by ε^5. But still

Exercise 11.11. Show that plugging the algorithm of Theorem 11.10 in the proof of the regularity lemma still produces an ε-regular partition in time $O(n^3)$.

One way to solve the above exercise is by checking the details of the proof. A more "abstract" way to see this is to observe that we are thus "proving" the regularity lemma for ε^5 instead of ε (so the partition will be of size $\mathrm{twr}(1/\varepsilon^{25})$ instead of $\mathrm{twr}(1/\varepsilon^5)$).

To prove of Theorem 11.10 we will apply an interesting equivalence between two quasi-random properties of bipartite graphs. We first make the following observation. Denote by P_5 the graph property that all pairs of vertices (x, y), possibly except εn^2 of them, satisfy $d(x, y) = (p^2 + \varepsilon)n$. We have the following:

Exercise 11.12. Show that $P_2 \iff P_5$. You can prove this either directly or by showing that P_5 is equivalent to P_4 and then using the equivalences we have already proved (try to do both).

The properties $P_1, ..., P_5$ of quasi-random graphs have each a natural variant that deals with quasi-random bipartite graphs. For example, suppose a bipartite graph G, on vertex sets A, B, with $|A| = |B| = n$, has pn^2 edges. We say that G satisfies property $\bar{P}_2$ if for every pair of vertex sets $A' \subseteq A, B' \subseteq B$ where $|A|, |B| \geq \varepsilon n$ we have $d(A', B') = p \pm \varepsilon$. This can be considered the "bipartite" version of property P_2 from the previous lecture (see Definition 10.9). More importantly, note that this property is simply the property of a bipartite graph being ε-regular (in the sense of the regularity lemma).

We can also define a bipartite analogue of property P_5. Suppose G is has vertex sets A, B, with $|A| = |B| = n$, and pn^2 edges. We say that G satisfies property $\bar{P}_5$ if all but at most εn^2 pairs of verticesx, $x, y \in A$, satisfy $d(x, y) = (p^2 \pm \varepsilon)n$. Then we have the following bipartite version of the equivalence between P_2 and P_5.

Exercise 11.13. Show that $\bar{P}_2 \iff \bar{P}_5$. In particular, if (A, B) is not ε-regular, then A contains $\varepsilon^4 n^2$ pairs of vertices x, y for which $d(x, y) \geq (d^2 + \varepsilon^4)n$, where $d = d(A, B)$.

The implication $\bar{P}_2 \implies \bar{P}_5$ is easy (do it!). The more interesting part is the other direction (which is the only direction we will need in the proof). To prove this direction one can check that all the arguments one needs to prove that $P_5 \implies P_2$ carry almost word for word to the bipartite setting. Another shorter way is to reduce the fact that $\bar{P}_5 \implies \bar{P}_2$ to the fact that $P_5 \implies P_2$.

Proof (Theorem 11.10): If G is not ε-regular, then by Exercise 11.13 the set A contains at least $\varepsilon^4 n^2$ pairs of vertices x, y satisfying $d(x, y) \geq (d^2 + \varepsilon^4)n$. It is obvious that such pairs can be found in $O(n^3)$

[11] Here and later in the proof, when we say polynomial, we mean polynomial in n. The dependence on ε would still be of tower-type since the size of the partition can be of this size. So one can think of ε as fixed and the input size being n.

time (of course, we might not find these pairs if A, B is ε-regular but we need not worry about this case.). We will now show how to use these $\varepsilon^4 n^2$ pairs in order to find the sets A', B'.

Assume first that A contains $2\varepsilon^5 n$ vertices x satisfying $d(x) \neq (d \pm \varepsilon^5)n$, and assume w.l.o.g. that $\varepsilon^5 n$ of them have degree at least $(d + \varepsilon^5)n$. Then the algorithm can return these vertices as the A' and as B' it can return B itself.

Assume now that there is $x \in A$ satisfying $d(x) = (d \pm \varepsilon^5)n$ and that there are at least $\varepsilon^5 n$ other vertices $x' \in A$ satisfying $d(x, x') \geq (d^2 + \varepsilon^4)n$. We claim that in this case the algorithm can return $N(x)$ as the set B' and the $\varepsilon^5 n$ vertices x' satisfying $d(x, x') \geq (d^2 + \varepsilon^4)n$ as the set A'. Indeed, note that[12] $|B'| \geq (d - \varepsilon^5)n \geq \varepsilon^5 n$, and $|A'| \geq \varepsilon^5 n$ so both sets are of the required size. As to $d(A', B')$, we have[13]

$$d(A', B') = \frac{e(A', B')}{|A'||B'|} \geq \frac{|A'|(d^2 + \varepsilon^4)n}{|A'|(d + \varepsilon^5)n} = \frac{d^2 + \varepsilon^4}{d + \varepsilon^5} \geq d + \varepsilon^5 \; .$$

Finally, observe that if none of the above two cases holds then A contains at most $(2\varepsilon^5)n$ vertices satisfying $d(x) \neq (d + \varepsilon^5)n$ and for any vertex that does satisfy $d(x) \neq (d + \varepsilon^5)n$ there are at most $\varepsilon^5 n$ vertices satisfying $d(x, x') \geq (d^2 + \varepsilon^4)n$. But this means that there are at most $(2\varepsilon^5)n \cdot n + n \cdot \varepsilon^5 n \leq \varepsilon^4 n^2$ pairs of vertices satisfying $d(x, x') \geq (d^2 + \varepsilon^4)n$ which is a contradiction. $\square$

11.3 Approximating MAX-CUT using the Regularity Lemma

We now give an application of the algorithmic version of the regularity lemma we obtain in the previous subsection. Our focus will be the MAX-CUT problem. In this problem the input is a graph G and the goal is (as the name suggests) to find a partition of $V(G)$ into two sets A, B so as to maximize the number of edges connecting A to B. This problem is known to be NP-hard so we try to find a good approximation. Observe that Claim 8.8 we previously proved shows that we can always find a partition which achieves at least half of the optimum (from now on let us use $OPT = OPT(G)$ for the optimal solution for G).

Using the Regularity Lemma we can obtain the following:

Theorem 11.14. *There is an $O(n^3)$ algorithm that given a graph G finds a partition A, B satisfying*

$$e(A, B) \geq OPT - \varepsilon n^2 \; .$$

Note that the above result is useless when G has $o(n^2)$ edges. On the other hand, when G has cn^2 edges, we know from Claim 8.8 that $OPT \geq \frac{1}{2}cn^2$ so applying the above theorem with $\frac{1}{2}\varepsilon c$ we can find a partition A, B satisfying $e(A, B) \geq (1 - \varepsilon)OPT$.

Proof (Theorem 11.14): It will be simpler to prove the result with error $4\varepsilon n^2$. We already know (via Theorem 11.10) how to find an ε-regular equipartition $V_1, ..., V_k$ of G in time $O(n^3)$. Therefore, we only need to show how to find the partition A, B in time $O(n^3)$ given $V_1, ..., V_k$. Note that we can disregard the edges inside the clusters since this changes OPT by at most εn^2.

We will now show that given the densities between all pairs of clusters V_i, V_j we can approximate OPT in constant time! So suppose we are given the densities $d_{i,j} = d(V_i, V_j)$. For every choice of $a_1 \in \{0, ..., 1/\varepsilon\}, ..., a_k \in \{0, ..., 1/\varepsilon\}$ let us set $x_i = \varepsilon a_i$ and define

$$e(x_1, ..., x_k) = \sum_{i<j} d_{ij} \left(x_i(1 - x_j) + (1 - x_i)x_j \right) |V_i||V_j| \; .$$

[12] Assuming that $d \geq 2\varepsilon^5$, otherwise the graph is sparse and will trivially be ε-regular anyway. In fact, already when $d \leq \varepsilon^3$ the graph is trivially ε-regular.

[13] Here we are also assuming that $\varepsilon < \varepsilon_0$. We can justify this assumption by simply replacing ε with ε_0 if this is not the case. The only affect this might have is that instead of having ε^5 in the statement of the Theorem we will get $c\varepsilon^5$ for some absolute constant $c = \varepsilon_0^5$.

Observe that we can interpret $e(x_1, ..., x_k)$ as $e(A, B)$ when A is obtained by picking $x_i|V_i|$ vertices from each of the sets V_i. This is of course just an approximation of $e(A, B)$, but it is easy to see that since $V_1, ..., V_k$ is ε-regular, that no matter how we pick the $x_i|V_i|$ vertices from each cluster V_i, we always have

$$|e(A, B) - e(x_1, ..., x_k)| \leq \varepsilon n^2 \ . \tag{11.5}$$

We thus claim that after computing $e(x_1, ..., x_k)$ for all possible $(1/\varepsilon)^k$ assignments, we can pick the assignment $x_1, ..., x_k$ that achieves the maximum, define a partition A, B using $x_1, ..., x_k$ as above, and return it as the output.

To show that we indeed have $e(A, B) \geq OPT - \varepsilon n^2$, let X, Y be the partition of $V(G)$ satisfying $e(X, Y) = OPT$. Now define a new partition X', Y' so that for every $1 \leq i \leq k$ the set $|X' \cap V_i| = \varepsilon a'_i|V_i|$, where a'_i is an integer. This partition can clearly be obtained from X, Y by "moving around" at most εn vertices. Since this can change the number of edges connecting the two sets by at most εn^2, we have

$$e(X', Y') \geq e(X, Y) - \varepsilon n^2 = OPT - \varepsilon n^2 \ .$$

Setting $x'_i = \varepsilon a'_i$ we now get from (11.5) that

$$e(x'_1, ..., x'_k) \geq e(X', Y') - \varepsilon n^2 \geq OPT - 2\varepsilon n^2 \ .$$

Since we defined (A, B) using an assignment that maximizes $e(x_1, ..., x_k)$ we have

$$e(x_1, ..., x_k) \geq e(x'_1, ..., x'_k) \geq OPT - 2\varepsilon n^2 \ .$$

Finally we get from another application of (11.5) that

$$e(A, B) \geq e(x_1, ..., x_k) - \varepsilon n^2 \geq OPT - 3\varepsilon n^2 \ ,$$

as needed. Recalling the error of εn^2 that resulted from disregarding the edges inside the clusters we get an approximation of $4\varepsilon n^2$. □

12 Twelfth lecture

12.1 Hypergraph Ramsey Numbers

Recall that $N = R(s, t)$ is the smallest number such that every 2 coloring of K_N contains either a red K_s or black K_t. We showed that $R(s, t) \leq \binom{s+t-2}{s-1}$ and in particular $2^{t/2} \leq R(t, t) \leq 4^t$. This result can be generalized to 3-graphs.

Theorem 12.1. *For every s, t there exists $N = R_3(s, t)$ such that every 2 coloring of K^3_N contains a red K^3_s or a black K^3_t.*

Proof: We show that

$$R_3(s, t) \leq R_2\left(R_3(s-1, t), R_3(s, t-1)\right) + 1 \ . \tag{12.1}$$

We begin by picking an arbitrary vertex v and let X denote the rest of the vertices. Define a 2-coloring of the complete graph on the vertex set X as follows; we color edge (i, j) using the color given to (v, i, j). By Ramsey's Theorem for graphs, this 2-coloring has either a vertex set X_1 that spans a red $K_{R_3(s-1,t)}$ or a vertex set X_2 that spans a black $K_{R_3(s,t-1)}$. Assume the former holds. By induction, the hypergraph induced by X_1 contains either black K^3_t (and we are done) or a red K^3_{s-1}. It is now easy to see that the way we defined the coloring of X guarantees that X_1 together with v gives a red K^3_s, thus completing the induction step for proving (12.1). The base cases can easily be checked, thus proving the theorem. □

Let us try to get a feeling of the type of bound we get from the above proof. Since $R(t,t) \geq 2^{t/2}$ we see that the recurrence relation in (12.1) satisfies

$$R_3(t,t) \geq R_2(R_3(t-1,t-1), R_3(t-1,t-1)) \geq 2^{R_3(t-1,t-1)/2}$$

implying that this bound grows like a tower of $\sqrt{2}$ of height t, which is a pretty weak. We now give a better bound based on an argument of Erdős and Rado.

Theorem 12.2. $R_3(t,t) \leq 2^{\binom{R_2(t,t)}{2}}$. *In particular* $R_3(t,t) \leq 2^{2^{4t}}$.

Proof: We try to imitate the first proof of Ramsey's Theorem for graphs we previously gave. We iteratively construct a 3-tuple (S_i, ϕ_i, X_i) where S_i and X_i are two disjoint vertex sets, $S_i = \{s_1, ..., s_i\}$, and ϕ_i is a 2-coloring of the pairs of vertices of S_i satisfying the following two key conditions: (1) For every $1 \leq a < b < c \leq i$ the edge (S_a, S_b, S_c) is colored $\phi_i(S_a, S_b)$, and (2) For every of $1 \leq a < b \leq i$ and $x \in X_i$ the edge (S_a, S_b, x) is also colored $\phi_i(S_a, S_b)$.

Suppose first that we have managed to construct such a 3-tuple with $i = R_2(t,t)$. Then by Ramsey's Theorem, the coloring ϕ_i of S_i (when viewed as a 2-coloring of the complete graph on $R_2(t,t)$ vertices) contains a monochromatic K_t. It is now easy to see that the first key property of ϕ_i guarantees that the vertices of this K_t also form a monochromatic K_t^3 in the hypergraph.

To construct the 3-tuples (S_i, ϕ_i, X_i) we start with $S_1 = \{v\}$, with v an arbitrary vertex, and $X_1 = V \setminus v$. Assuming we defined (S_i, ϕ_i, X_i) we now define $(S_{i+1}, \phi_{i+1}, X_{i+1})$. Pick an arbitrary vertex from X_i, remove it from X_i and add it to S_i. This vertex is going to be s_{i+1}. We now need to define the new set X_{i+1} and the new coloring ϕ_{i+1}. The new coloring ϕ_{i+1} will agree with ϕ_i on all pairs (S_a, S_b) with $1 \leq a < b \leq i$. So we just need to color the pairs (S_a, S_{i+1}) with $1 \leq a \leq i$. The set X_i clearly contains a subset X_i^1 of size at least $|X_i|/2$ so that all the triples (s_1, s_{i+1}, x), with $x \in X_i^1$, are colored with the same color. If this color is red we color (s_1, s_{i+1}) red, otherwise we color it black. In a similar manner, the set X_i^1 clearly contains a subset X_i^2 of size at least $|X_i^1|/2$ so that all the triples (s_2, s_{i+1}, x), with $x \in X_i^2$, are colored with the same color. If this color is red we color (s_2, s_{i+1}) red, otherwise we color it black. We continue in this manner and set $X_{i+1} = X_i^i$. It is easy to see that $(S_{i+1}, \phi_{i+1}, X_{i+1})$ has the required properties. Finally, note that when constructing (S_i, ϕ_i, X_i), in each of the $\binom{i}{i}$ iterations we were forced to reduce the size of X_1 by a factor of at most 2. Therefore, if the initial hypergraph is of size at least $2^{\binom{R_2(t,t)}{2}}$ we are guaranteed that the process will be able to go on for $i = R_2(t,t)$ iterations. $\square$

Somewhat embarrassingly, the best lower for $R_3(t,t)$ is the one one gets from the obvious random construction as the following exercise suggests.

Exercise 12.3. Show that $R_3(t,t) \geq 2^{t^2/6}$.

One can easily adapt the proof of our previous upper bound $R_3(t,t) \leq 2^{\binom{R_2(t,t)}{2}}$ and obtain the following more general recursive relation.

Exercise 12.4. Show that for every $k \geq 3$ we have $R_k(t,t) \leq 2^{\binom{R_{k-1}(t,t)}{k-1}}$.

We see that any upper bound for R_k can be "lifted" to R_{k+1} with the price of one extra exponent. As the following theorem shows, the same holds also for lower bounds, but only for $k \geq 3$.

Theorem 12.5 (Step-up Lemma)**.** *The following holds for every $k \geq 3$; if there is a 2-coloring of K_m^k with no monochromatic K_t then there is a 2-coloring of $K_{2^m}^{k+1}$ with no monochromatic K_{2t+k}, that is, $R_{k+1}(2t+k, 2t+k) \geq 2^m$.*

Proof: We prove the result for $k = 3$ and leave the general case as an exercise. Assuming we have a 2-coloring of K^3_m with no monochromatic K^3_{t-1} we define a 2-coloring of $K^4_{2^m}$ with no monochromatic K^4_{2t-1}.

We think of every vertex of $K^4_{2^m}$ as a binary string of length m. For such strings x, y we define $\delta(x, y)$ to be the largest index where the strings differ. We also assume that the strings/vertices are ordered lexicographically. We will rely on the following facts that are easy to verify.

1. If $x < y < z$ then $\delta(x, y) \neq \delta(y, z)$.

2. If $x < y < z$ then $\delta(x, z) = \max\{\delta(x, y), \delta(y, z)\}$.

Note that the second fact above implies that more generally, if $x_1 < \ldots < x_l$ then

$$\delta(x_1, x_l) = \max_{1 \leq i \leq l-1} \delta(x_i, x_{i+1}) \tag{12.2}$$

We now show how to color $K^4_{2^m}$. Given $x_1 < x_2 < x_3 < x_4$, set $\delta_1 = \delta(x_1, x_2), \delta_2 = \delta(x_2, x_3), \delta_3 = \delta(x_3, x_4)$. We use the following rules:

1. If $\delta_1 < \delta_2 < \delta_3$ we color (x_1, x_2, x_3, x_4) using the color given to $(\delta_1, \delta_2, \delta_3)$.

2. If $\delta_1 > \delta_2 > \delta_3$ we color (x_1, x_2, x_3, x_4) using the color given to $(\delta_1, \delta_2, \delta_3)$.

3. If $\delta_1 > \delta_2 < \delta_3$ we color (x_1, x_2, x_3, x_4) black.

4. If $\delta_1 < \delta_2 > \delta_3$ we color (x_1, x_2, x_3, x_4) red.

Assume $x_1 < \ldots < x_{2t-1}$ is a monochromatic K^4_{2t-1}, and suppose it is red (the black case is identical). Consider $\delta_1, \delta_2, \ldots, \delta_{2t-2}$. By coloring rules 3 and 4, there is at most one consecutive triplet which is not monotone. As the sequence is of length $2t - 2$, it must contain a consecutive monotone sub-sequence of length $t - 1$. Assume, without loss of generality that it is of the form $\delta_1 < \delta_2 < \ldots < \delta_{t-1}$ and consider $x_1, \ldots, x_t$. We get that for every $1 \leq i \leq t - 3$, the edge $(\delta_i, \delta_{i+1}, \delta_{i+2})$ is colored red. Consider now a general triple $(\delta_i, \delta_j, \delta_k)$ and take the 4-edge $(x_i, x_{i+1}, x_{j+1}, x_{k+1})$. We get from the monotonicity of the δ's and (12.3) that $\delta(x_{i+1}, x_{j+1}) = \delta_j$ and that $\delta(x_{j+1}, x_{k+1}) = \delta_k$. Since $(x_i, x_{i+1}, x_{j+1}, x_{k+1})$ was colored red and $\delta_i < \delta_j < \delta_k$, this means that $(\delta_i, \delta_j, \delta_k)$ was also colored red, implying that $\delta_1, \delta_2, \ldots, \delta_{t-1}$ form a red K^3_{t-1}, which is a contradiction. □

Let us define $\mathrm{twr}_1(x) = x$ and $\mathrm{twr}_k(x) = 2^{\mathrm{twr}_{k-1}(x)}$. So, for example, $\mathrm{twr}_3(x)$ is the function 2^{2^x}. It follows from the fact that $R_2(t, t) \leq 2^{2t}$ and the general reduction from R_{k+1} to R_k that $R_k(t, t) \leq \mathrm{twr}_k((4 + o(1))t)$. Furthermore, if one could improve the upper bound for 3-graphs by showing that $R_3(t, t) \leq 2^{\mathrm{poly}(t)}$ then we would get that $R_k(t, t) \leq \mathrm{twr}_{k-1}(\mathrm{poly}(t))$. However, it is widely believed that $R_3(t, t)$ is indeed double exponential. It also follows from the Step-Up Lemma that showing that $R_3(t, t)$ is indeed double-exponential, that is, showing that $R_3(t, t) \geq 2^{2^{ct}}$, will give a tight bound for general k, that is, imply that $R_k(t, t) \geq \mathrm{twr}_k(ct)$. The bottom line is that determining $R_3(t, t)$ would thus determine $R_k(t, t)$ for all k.

Somewhat surprisingly, for 4 colors we can obtain rather tight bounds for $R_k(t, t, t, t)$. The upper bound follows by adapting the above proof for 2-colors and is thus left as an exercise.

Exercise 12.6. Show that $R_3(t, t, t, t) \leq 2^{2^{ct}}$ and that more generally $R_k(t, t, t, t) \leq \mathrm{twr}_k(O(t))$

We now show that for 4 colors we can obtain a matching lower bound for $k = 3$. This lower bound can then be lifted to larger k using a version of the stepup lemma we gave earlier for 2-colors.

Theorem 12.7. *Suppose there is a 2-coloring of K_m with no monochromatic K_{t-1}. Then $R_3(t,t,t,t) \geq 2^m$. In particular, we have $R_3(t,t,t,t) \geq 2^{2^{t/2}}$.*

Proof: We define a 4-coloring of $K^3_{2^m}$ without a monochromatic K^3_t. As in the last proof, we think of the vertices as binary strings of length m, and define δ as before. We define the coloring as follows: given $x < y < z$, set $\delta_1 = \delta(x,y), \delta_2 = \delta(y,z)$, and color the edge (x,y,z) as follows:

1. If $\delta_1 < \delta_2$ and (δ_1, δ_2) is red (in the assumed K_m) then color (x,y,z) with color A.
2. If $\delta_1 < \delta_2$ and (δ_1, δ_2) is black then color (x,y,z) with color B.
3. If $\delta_1 > \delta_2$ and (δ_1, δ_2) is red then color (x,y,z) with color C.
4. If $\delta_1 > \delta_2$ and (δ_1, δ_2) is black then color (x,y,z) with color D.

Assume that $x_1 < x_2 < \ldots < x_t$ form a monochromatic K^3_t whose color is A (the other 3 cases are identical). Define $\delta_i = \delta(x_i, x_{i+1})$ for every $1 \leq i \leq t-1$. We will now show that $\{\delta_i\}_{1 \leq i \leq t-1}$ forms a red clique of size $t-1$ in the original graph K_m, which will be a contradiction. First note that since all triples x_i, x_{i+1}, x_{i+2} are colored A, we get

$$\delta_1 < \delta_2 < \delta_3 < \ldots < \delta_{t-1} \tag{12.3}$$

so these vertices are indeed distinct. We also get from the definition of the coloring, that for every $1 \leq i \leq t-2$ the edge (δ_i, δ_{i+1}) is colored red. For every $i < j$ consider the edge (x_i, x_{i+1}, x_{j+1}). We now get from (12.2) and (12.3) that $\delta(x_{i+1}, x_{j+1}) = \delta(x_j, x_{j+1}) = \delta_j$. Since (x_i, x_{i+1}, x_{j+1}) is colored A, we get that (δ_i, δ_j) must be colored red as well. We thus get that $\delta_1, ..., \delta_{t-1}$ form a red K_{t-1}. $\square$

13 Notations

$d(x)$	The degree of vertex x
$\delta(G)$	$\min_{x \in V(G)} d(x)$
$e(G)$	$\lvert E(G) \rvert$ – the number of vertices in G
$ex(n,H)$	The maximal number of edges for a H-free graph on n vertices
K_n	The complete graph on n vertices
$K_{n_1,\ldots,n_r}$	The complete r-partite graph with partitions of size $n_1, n_2, \ldots, n_r$
K^s_r	r-graph with a clique of size s. Sometimes written with super- and subscript reversed.

References

F.A. Behrend. On sets of integers which contain no three terms in arithmetical progression. *Proc. Nat. Acad. Sci. USA*, 32:331–332, 1946.

B. Bollobas and A. Thomason. Highly linked graphs. *Combinatorica*, 16(3):313–320, 1996.

JA Bondy. Pancyclic graphs I. *Journal of Combinatorial Theory, Series B*, 11(1):80–84, 1971.

F.R.K. Chung, R.L. Graham, and R.M. Wilson. Quasi-random graphs. *Combinatorica*, 9(4):345–362, 1989.

K.F. Roth. On certain sets of integers. *Journal of the London Mathematical Society*, 1(1):104, 1953.

I.Z. Ruzsa and E. Szemerédi. Triple systems with no six points carrying three triangles. *Combinatorics (Keszthely, 1976), Coll. Math. Soc. J. Bolyai*, 18:939–945, 1978.

E. Szemerédi. On sets of integers containing no four elements in arithmetic progression. *Acta Mathematica Hungarica*, 20(1):89–104, 1969.

E. Szemerédi. On sets of integers containing no k elements in arithmetic progression. *Acta Arith*, 27 (585):199–245, 1975.