Bounding the number of odd paths in planar graphs via convex optimization

Asaf Cohen Antonir*   Asaf Shapira †

Abstract

Let $N_{\mathcal{P}}(n,H)$ denote the maximum number of copies of H in an n vertex planar graph. The problem of bounding this function for various graphs H has been extensively studied since the 70's. A special case that received a lot of attention recently is when H is the path on $2m+1$ vertices, denoted P_{2m+1}. Our main result in this paper is that

$$N_{\mathcal{P}}(n,P_{2m+1}) = O(m^{-m}n^{m+1}) \ .$$

This improves upon the previously best known bound by a factor e^m, which is best possible up to the hidden constant, and makes a significant step towards resolving conjectures of Gosh et al. and of Cox and Martin. The proof uses graph theoretic arguments together with (simple) arguments from the theory of convex optimization.

1 Introduction

In this paper we study the following extremal problem: given a fixed graph H, what is the maximum number of copies of H that can be found in an n vertex planar graph? We denote this maximum by $N_{\mathcal{P}}(n,H)$. The investigation of this problem was initiated by Hakimi and Schmeichel [9] in the 70's. They considered the case when H is a cycle of length m, denoted C_m. They determined $N_{\mathcal{P}}(n,C_3)$ and $N_{\mathcal{P}}(n,C_4)$ exactly, and for general $m \geq 3$ proved that $N_{\mathcal{P}}(n,C_m) = \Theta(n^{\lfloor m/2 \rfloor})$. Following this result, Alon and Caro [1] determined $N_{\mathcal{P}}(n,K_{2,m})$ exactly for all m, where $K_{2,m}$ is the complete 2-by-m bipartite graph. In a series of works [3, 8, 12], which culminated with a recent paper of Huynh, Joret and Wood [10], the asymptotic value of $N_{\mathcal{P}}(n,H)$ was determined up to a constant factor (depending on H) for every fixed H.

The next natural question following the result of [10] is to determine the asymptotic growth of $N_{\mathcal{P}}(n,H)$ up to[1] $1+o(1)$, or more ambitiously, to determine its exact value. This line of research was initiated by Győri, Paulos, Salia, Tompkins and Zamora [6, 8], who showed that for large enough n we have $N_{\mathcal{P}}(n,P_4) = 7n^2 - 32n + 27$ and $N_{\mathcal{P}}(n,C_5) = 2n^2 - 10n + 12$, where P_m denotes the

*School of Mathematical Sciences, Tel Aviv University, Tel Aviv, 6997801, Israel. Email: asafc1@tauex.tau.ac.il

†School of Mathematics, Tel Aviv University, Tel Aviv 69978, Israel. Email: asafico@tau.ac.il. Supported in part by ERC Consolidator Grant 863438 and NSF-BSF Grant 20196.

[1] We use the standard notation $o(1)$ to denote a quantity tending to 0 when n tends to infinity and H is fixed. Similarly, when we write $o(n^k)$ we mean $o(1) \cdot n^k$.

path with m vertices (and $m-1$ edges). We note that the result of Alon and Caro [1] implies that $N_{\mathcal{P}}(n, K_{1,2}) = N_{\mathcal{P}}(n, P_3) = n^2 + 3n - 16$. Addressing the problem of finding the asymptotic value of $N_{\mathcal{P}}(n, P_m)$ up to $1 + o(1)$, Ghosh, Győri, Martin, Paulos, Salia, Xiao and Zamora [4], showed that $N_{\mathcal{P}}(n, P_5) = (1 + o(1))n^3$. They also raised the following conjecture[2] regarding the asymptotic value of $N_{\mathcal{P}}(n, P_{2m+1})$ for arbitrary $m \geq 2$:

$$N_{\mathcal{P}}(n, P_{2m+1}) = (4m^{-m} + o(1))n^{m+1} \, . \tag{1}$$

We note that the lower bound in (1) is easy. Indeed, start with a cycle of length $2m$, and then replace every second vertex with an independent set consisting of $(n-m)/m$ vertices, each with the same neighborhood as the original vertex it replaced.

In a very recent paper, Cox and Martin [2] introduced an analytic approach for proving (1). They showed that

$$N_{\mathcal{P}}(n, P_{2m+1}) \leq (\rho(m)/2 + o(1))n^{m+1} \, , \tag{2}$$

where $\rho(m)$ is the solution to a certain convex optimization problem, which we define precisely in Section 3. They further conjectured that

$$\rho(m) \leq 8m^{-m} \, , \tag{3}$$

which, if true, implies (1). In the same paper, they verified their conjecture for $m = 3$ by showing that $\rho(3) = 8/27$, which confirms (1) for $m = 3$. Using the same approach they also improved the known asymptotic value of $N_{\mathcal{P}}(n, P_{2m+1})$ by showing that

$$N_{\mathcal{P}}(n, P_{2m+1}) \leq \left(\frac{1}{2 \cdot (m-1)!} + o(1) \right) n^{m+1} \, .$$

Note that this bound is roughly e^m larger than the one conjectured in (1).

Our main result in this paper, Theorem 1.1 below, makes a significant step towards the resolution of the Cox–Martin and Gosh et.al. conjectures, by establishing (3) up to an absolute constant.

Theorem 1.1. *There is an absolute constant C so that for every fixed $m \geq 2$ and large enough n, we have*

$$N_{\mathcal{P}}(n, P_{2m+1}) \leq C m^{-m} n^{m+1} \, .$$

As noted after (1), the above bound is best possible up to the value of C. Furthermore, as can be seen in the proof of Theorem 1.1, the constant C we obtain is 10^4 (which can certainly be improved).

1.1 Related work and paper overview

In addition to studying $N_{\mathcal{P}}(n, P_{2m+1})$, Cox and Martin [2] also introduced an analytic method for bounding the maximum number of even cycles in planar graphs. Similar to the case of odd paths discussed above, they showed that $N_{\mathcal{P}}(n, C_{2m}) \leq (\beta(C_m) + o(1))n^m$, where $\beta(C_m)$ is an optimization problem, similar to the one we study in Section 2. They conjectured that $\beta(C_m) = m^{-m}$, a bound which implies by their framework that $N_{\mathcal{P}}(n, C_{2m}) \leq (1 + o(1))(n/m)^m$. Observe that the example

[2] They also conjectured that the second order term is $O(n^m)$.

we mentioned after (1) shows that this bound is best possible. Towards their conjecture, Cox and Martin [2] proved that $\beta(C_m) \leq 1/m!$. Using the ideas in this paper, one can significantly improve this bound. In particular, using Lemma 2.5 in Section 2, it is not hard to show that for some absolute constant C, we have

$$\beta(C_m) \leq Cm^{-m} \ . \tag{4}$$

In an independent work, Lv, Győri, He, Salia, Tompkins and Zhu [11] confirmed the conjecture of Cox and Martin by showing that one can in fact obtain $C = 1$ in (4). We thus do not include the proof of (4).

We should point that the reason why studying $N_{\mathcal{P}}(n, P_{2m+1})$ appears to be much harder than $N_{\mathcal{P}}(n, C_{2m})$ is that as opposed to $\beta(C_m)$, which is an optimization problem involving a single graph, $\rho(m)$ is an optimization problem which involves several multigraphs. To overcome this difficulty we first study in Section 2 an optimization problem, denoted $\beta(P_m)$, which is the analogue of $\beta(C_m)$ for the setting of P_m. The main advantage of first studying $\beta(P_m)$ is that it allows us to employ a weight shifting argument, which does not seem to be applicable to $\rho(m)$. Our main result in that section is a nearly tight bound for $\beta(P_m)$. However, as opposed to the case of $N_{\mathcal{P}}(n, C_{2m})$, a bound for $\beta(P_m)$ does not immediately translate into a bound for $N_{\mathcal{P}}(n, P_{2m+1})$. Hence, in Section 3 we show how one can transfer any bound for $\beta(P_m)$ into a bound for $\rho(m)$, thus proving Theorem 1.1. To this end we use simple arguments from the theory of convex optimization, along with the fact that $\rho(m)$ is a low degree polynomial.

The key lemmas leading to the proof of Theorem 1.1 are Lemmas 2.2 and 3.2 for which we obtain bounds that are optimal up to constant factors. Moreover, if one can improve these bounds to the optimal conjectured ones, then this will give the conjectured inequality (1). We believe that with more care, it is possible to use the ideas in this paper to improve the bound for $\beta(P_m)$ in Lemma 2.2 to the conjectured one. In contrast, because of the complex structure of $\rho(m)$, it seems that in order to improve the bound in Lemma 3.2 to the conjectured bound, a new idea is needed.

2 A variant of $\rho(m)$

Our goal in this section is to prove Lemma 2.2 regarding the optimization problem $\beta(P_m)$. This lemma will be used in the next section in the proof of Theorem 1.1. The proof of Lemma 2.2 will employ a subtle weight shifting argument. We first recall several definitions from [2]. In what follows we write $[n]$ to denote the set $\{1, \ldots, n\}$ and K_n to denote the complete graph on $[n]$.

Definition 2.1. *Let $n > 0$ be an integer, and let μ be a probability measure on the edges of K_n.*

1. *For any $x \in [n]$ we define the weighted degree of x to be*

$$\bar{\mu}(x) = \sum_{y \in [n] \setminus \{x\}} \mu(x, y) \ .$$

2. *For any subgraph $H \subseteq K_n$ we define the wight of H to be*

$$\mu(H) = \prod_{e \in E(H)} \mu(e) \ .$$

3. *For any graph with no isolated vertices H define*

$$\beta(\mu; H) = \sum_{H' \in \mathbf{C}(H,n)} \mu(H') ,$$

where $\mathbf{C}(H,n)$ is the set of all copies of H in K_n. Further, we define

$$\beta(H) = \sup_{\mu} \beta(\mu; H) ,$$

where the supremum is taken over all probability measures μ on the edges of $K_{n'}$ for some n'.

Intuitively, the function $\beta(\mu; H)$ is the probability of hitting a copy of H if $|E(H)|$ independent edges were chosen according to μ.

Lemma 2.2. *For any integer $m \geq 2$ we have*

$$\beta(P_m) \leq \frac{20}{m^{m-2}} .$$

We remark that this lemma is optimal up to the constant factor 20. To see this, consider the uniform distribution over the edges of C_m, which shows that $\beta(P_m) \geq 1/m^{m-2}$. It seems reasonable to conjecture that $\beta(P_m) = 1/m^{m-2}$.

The key step in the proof of Lemma 2.2 is Lemma 2.5 below. To state this lemma, we first need the following definitions.

Definition 2.3. *For every $k, \ell \geq 0$ we define $P_{(k,\ell)}$ to be a disjoint union of P_{k+1} and $P_{\ell+1}$.*

From now on, we will not only deal with probability measures but also with bounded measures. Therefore, we will frequently write *measure* to denote a bounded measure. Moreover, for a measure μ we will denote its total mass by $w(\mu)$.

Definition 2.4. *Suppose μ is a measure on the edges of K_n and $s, t \geq 0$. Define*

$$\beta^*(\mu; P_{(s,t)}) = \sum_{P \in \mathbf{C}^*(P_{(s,t)},n)} \mu(P) ,$$

where $\mathbf{C}^*(P_{(s,t)}, n)$ is the set of copies of $P_{(s,t)}$ in K_n where the path of length s starts with the vertex n, and the path of length t starts with the vertex 1. Further, for every $w > 0$ we define

$$\beta^*_{w,n}(P_{(s,t)}) = \sup_{\mu} \beta^*(\mu; P_{(s,t)}) ,$$

where the supremum is taken over all measures μ on the edges of K_n with $w(\mu) = w$.

We remark that for any measure μ on the edges of K_n, we have $\beta^*(\mu; P_{(0,0)}) = 1$. This is because $\mathbf{C}^*(P_{(0,0)}, n)$ consist of a single graph, the independent set $I_2 = \{1, n\}$, and because $\mu(I_2) = 1$. This clearly implies that $\beta^*_{w,n}(P_{(0,0)}) = 1$ for every w and n.

Lemma 2.5. *For every $0 \leq \ell \leq m \leq n$ we have*

$$\beta^*_{1,n}(P_{(\ell,m-\ell)}) \leq \frac{1}{m^m} .$$

Claim 2.6. *Suppose that t is a non-negative integer, s, n are positive integers, and $w \geq 0$. Then, there exists a measure μ on the edges of K_n with $w(\mu) = w$, satisfying:*

1. $\beta^*(\mu; P_{(s,t)}) = \beta^*_{w,n}(P_{(s,t)})$,
2. *and for all $q \neq n-1$ we have $\mu(q, n) = 0$.*

Proof. The main idea in the proof is the introduction of the notion of a w-useful measure. We say that a measure μ on the edges of K_n with $w(\mu) = w$ is w-optimal if

$$\beta^*(\mu; P_{(s,t)}) = \beta^*_{w,n}(P_{(s,t)}) \ .$$

We further say that μ is w-useful if μ is w-optimal and

$$\max_{k \in [n-1]} \mu(n, k) = \sup_{\eta, k} \eta(n, k) \ ,$$

where the supremum is taken over all $k \in [n-1]$ and all measures η which are w-optimal. Let us see why such a w-useful measure exists. Note that there is a natural bijection between measures μ with $w(\mu) = w$, and vectors in the simplex $\Delta = \{x \in \mathbb{R}^{\binom{n}{2}} : x_i \geq 0 \text{ and } \sum_{i=1}^{\binom{n}{2}} x_i = w\}$. Thus, to show that a w-useful measure exists we think of μ as a vector in Δ. Recalling that

$$\beta^*(\mu; P_{(s,t)}) = \sum_{P \in \mathbf{C}^*(P_{(s,t)}, n)} \mu(P) = \sum_{P \in \mathbf{C}^*(P_{(s,t)}, n)} \prod_{e \in E(P)} \mu(e) \ ,$$

we see that $\beta^*(\mu; P_{(s,t)})$ is an $\binom{n}{2}$-variate polynomial, with variables $\mu(e)$ for all $e \in E(K_n)$. Under these notations, w-optimal measures are maximal points of the polynomial $\beta^*(\mu; P_{(s,t)})$ in Δ. Since Δ is compact and $\beta^*(\mu; P_{(s,t)})$ is continuous, we deduce that O_w, the set of all w-optimal measures, is non-empty. Moreover, O_w is a compact set, since it is closed (as the preimage of a closed set under the continuous function $\beta^*(\mu; P_{(s,t)})$) and bounded (as it is contained in Δ). Setting $f(\mu) = \max_{k \in [n-1]} \mu(n, k)$, we find that μ is a w-useful measure if and only if it is a maximal point of f within O_w. Since O_w is compact and f is continuous, a w-useful measure exists.

We now prove that the existence of w-useful measures implies the claim. Indeed, let μ be a w-useful measure. Assume with out loss of generality[3] that $\mu(n-1, n) \geq 0$ is maximal among all $\mu(k, n)$. We claim that μ is as required. The first condition follows immediately from the fact that any w-useful measure is also w-optimal. Assume towards contradiction that the second condition fails, that is, that there exists a $q \neq n-1$ with $\mu(q, n) > 0$. We will now show that there is a measure μ' satisfying $w(\mu') = w$ which will either contradict the fact that μ is w-optimal or the fact that it is w-useful.

We define μ' as follows: We first set $\mu'(e) = \mu(e)$ for every edge other than the two edges $\{n-1, n\}$ and $\{q, n\}$. Define W_q to be the weight (under μ) of all copies of $P_{(s-1,t)}$, not containing n, such that the path of length $s-1$ starts with q, and the path of length t starts with 1. Define W_{n-1} analogously. Then, we define

$$\mu'(n, n-1) = \begin{cases} \mu(n-1, n) + \mu(q, n) & \text{if } W_{n-1} \geq W_q \ , \\ 0 & \text{else} \ , \end{cases}$$

[3] If this is not the case, we can permute the vertices and end up with such measure.

and

$$\mu'(q,n) = \begin{cases} 0 & \text{if } W_{n-1} \geq W_q \ , \\ \mu(q,n) + \mu(n-1,n) & \text{else} \ . \end{cases}$$

To see that we indeed get a contradiction, assume first that $W_{n-1} \geq W_q$. Since a copy of $P_{(s,t)}$ uses at most one of the edges $\{n-1, n\}$ and $\{q, n\}$, decreasing the value of $\{q, n\}$ by some ε while increasing that of $\{n-1, n\}$ by the same ε increases the total weight of copies of $P_{(s,t)}$ by $\varepsilon(W_{n-1} - W_q)$. We thus infer that

$$\beta^*(\mu'; P_{(s,t)}) = \beta^*(\mu; P_{(s,t)}) + \mu(q,n)(W_{n-1} - W_q) \geq \beta^*(\mu; P_{(s,t)}) \ .$$

Since $\mu'(n, n-1) > \mu(n, n-1)$ we see that μ' witnesses the fact that μ is not w-useful. If on the other hand $W_q > W_{n-1}$, then

$$\beta^*(\mu'; P_{(s,t)}) = \beta^*(\mu; P_{(s,t)}) + \mu(n-1,n)(W_q - W_{n-1}) > \beta^*(\mu; P_{(s,t)}) = \beta^*_{w,n}(P_{(s,t)}) \ ,$$

so μ' witnesses the fact that μ is not w-optimal. ■

Claim 2.7. *Suppose s, t are non-negative integers, n is a positive integer, and $w \geq 0$. Then, there are $w_1, \ldots, w_s \geq 0$ such that $\sum_{i=1}^s w_i \leq w$ and such that*

$$\beta^*_{w,n}(P_{s,t}) \leq \beta^*_{w',n-s}(P_{0,t}) \cdot \prod_{i=1}^{s} w_i \ ,$$

where $w' = w - \sum_{i=1}^s w_i$.

Proof. First, if $s + t + 1 \geq n$ then the claim is trivial, as $\mathbf{C}^*(P_{(s,t)}, n) = \emptyset$. So we assume for the rest of the proof that $s + t + 2 \leq n$. Let μ_0 be a measure on the edges of K_n as guaranteed by Claim 2.6. Since $\beta^*_{w,n}(P_{(s,t)}) = \beta^*(\mu_0; P_{(s,t)})$, it is enough to prove that there are $w_1, \ldots, w_s \geq 0$ such that $\sum_{i=1}^s w_i \leq w$ and

$$\beta^*(\mu_0; P_{(s,t)}) \leq \beta^*_{w',n-s}(P_{0,t}) \cdot \prod_{i=1}^{s} w_i \ , \tag{5}$$

where $w' = w - \sum_{i=1}^s w_i$. We define inductively a sequence of reals $w_1, \ldots w_k \geq 0$ with $\sum_{i=1}^k w_i \leq w$, along with measures $\mu_1, \ldots, \mu_k$ on the edges of $K_{n-1}, \ldots, K_{n-k}$, respectively, such that the following holds for all $1 \leq j \leq s$, where we set $w'_j = w - \sum_{i=1}^j w_i$:

(i) $w(\mu_j) = w'_j$,

(ii) $w_j = \mu_{j-1}(n-j+1, n-j)$,

(iii) for all $t \in [n-j-2]$ we have $\mu_j(n-j, t) = 0$,

(iv) $\beta^*(\mu_j; P_{(s-j,t)}) = \beta^*_{w'_j,n-j}(P_{(s-j,t)})$, and

(v) $\beta^*(\mu_{j-1}; P_{(s-j+1,t)}) \leq w_j \cdot \beta^*(\mu_j; P_{(s-j,t)})$.

Indeed, assuming $w_1, \ldots, w_j$ and $\mu_1, \ldots, \mu_j$ have already been chosen, we now choose w_{j+1} and μ_{j+1}. We first set $w_{j+1} = \mu_j(n-j, n-j-1) \geq 0$ so that the second condition holds. Further, set $\mu'_{j+1} = \mu_j|_{n-j-1}$, the restriction of μ_j to the edges of K_{n-j-1}. Observe that by the induction hypothesis on μ_j, we have $\mu_j(n-j, t) = 0$ for all $t \neq n-j-1$. Hence

$$w(\mu'_{j+1}) = w(\mu_j) - \sum_k \mu_j(n-j, k) = w(\mu_j) - w_{j+1} = w'_{j+1} ,$$

and

$$\beta^*(\mu_j; P_{(s-j,t)}) = w_{j+1} \cdot \beta^*(\mu'_{j+1}; P_{(s-j-1,t)}) \leq w_{j+1} \cdot \beta^*_{w'_{j+1}, n-j-1}(P_{(s-j-1,t)}) . \tag{6}$$

Let μ_{j+1} be the measure given by Claim 2.6 applied with $P_{(s-j-1,t)}$ and total mass w'_{j+1}. We claim that μ_{j+1} satisfies the inductive properties. The fact that it satisfies the first condition is immediate from its definition. To see that μ_{j+1} satisfies the last three conditions, note that by Claim 2.6 the measure μ_{j+1} satisfies

$$\beta^*(\mu_{j+1}; P_{(s-j-1,t)}) = \beta^*_{w'_{j+1}, n-j-1}(P_{(s-j-1,t)}) , \tag{7}$$

and $\mu_{j+1}(n-j-1, t) = 0$ for all $t \neq n-j-2$. Finally, combining (6) and (7) we obtain

$$\beta^*(\mu_j; P_{(s-j,t)}) \leq w_{j+1} \cdot \beta^*(\mu_{j+1}; P_{(s-j-1,t)}) ,$$

thus verifying the last three properties. Repeatedly applying property (v) we deduce that

$$\beta^*(\mu_0; P_{(s,t)}) \leq \beta^*(\mu_s; P_{(0,t)}) \cdot \prod_{i=1}^{s} w_i .$$

Since $\beta^*(\mu_s; P_{(0,t)}) = \beta^*_{w'_s, n-s}(P_{(0,t)}) = \beta^*_{w', n-s}(P_{(0,t)})$ (by property (iv) and the definition of w') we have thus proved (5) and the proof is complete. ■

We now use Claim 2.7 to prove Lemma 2.5.

Proof of Lemma 2.5. Claim 2.7 applied with $s = \ell, t = m - \ell$ and with $w = 1$ asserts that there are $w_1, \ldots, w_\ell \geq 0$ such that $\sum_{i=1}^{\ell} w_i \leq 1$ and such that

$$\beta^*_{1,n}(P_{\ell, m-\ell}) \leq \beta^*_{w', n-\ell}(P_{0, m-\ell}) \cdot \prod_{i=1}^{\ell} w_i , \tag{8}$$

where $w' = 1 - \sum_{i=1}^{\ell} w_i$. Clearly, for all integers s, t, k and $w \geq 0$ we have $\beta^*_{w,k}(P_{(s,t)}) = \beta^*_{w,k}(P_{(t,s)})$. Hence, using Claim 2.7 with $s = m - \ell, t = 0$ and with $w = w'$, we obtain a sequence $w_{\ell+1}, \ldots, w_m$ of non-negative reals, such that $\sum_{i=\ell+1}^{m} w_i \leq w'$ and such that

$$\beta^*_{w', n-\ell}(P_{0, m-\ell}) = \beta^*_{w', n-\ell}(P_{m-\ell, 0}) \leq \beta^*_{w'', n-m}(P_{(0,0)}) \cdot \prod_{i=\ell+1}^{m} w_i = \prod_{i=\ell+1}^{m} w_i , \tag{9}$$

where $w'' = w' - \sum_{i=\ell+1}^{m} w_i$, and we used the fact that $\beta^*_{w'',n-m}(P_{(0,0)}) = 1$ (see the remark after Definition 2.4). Combining (8) and (9), we infer that there are $w_1, \ldots, w_m \geq 0$ with $\sum_{i=1}^{m} w_i \leq 1$ such that

$$\beta^*_{1,n}(P_{\ell,m-\ell}) \leq \prod_{i=1}^{m} w_i \leq \left(\frac{\sum_{i=1}^{m} w_i}{m} \right)^m \leq \frac{1}{m^m} \ ,$$

where the second inequality is the AM-GM inequality, and the last inequality follows from the properties of the sequence $w_1, \ldots, w_m$. ■

To deduce Lemma 2.2 from the above claims, we recall a definition and a lemma from Cox and Martin [2] which we specialize here to the case of P_m.

Definition 2.8. *For an integer n, we denote by* $\mathrm{Opt}(n; H)$ *the set of all probability measures μ on the edges of K_n satisfying*

$$\beta(\mu; P_m) = \sup_{\eta} \beta(\eta; P_m) \ ,$$

where the supremum is taken over all probability measures η on the edges of K_n.

Lemma 2.9 (Lemma 4.5 in [2])**.** *For every $n \geq m \geq 2$ and $\mu \in \mathrm{Opt}(n; P_m)$, we have the following for all $x \in [n]$*

$$\bar{\mu}(x) \cdot (m-1) \cdot \beta(\mu; P_m) = \sum_{\substack{P \in \mathbf{C}(P_m, n) \\ V(P) \ni x}} \deg_P(x) \mu(P) \ .$$

Proof of Lemma 2.2. Suppose $n \geq m$ and take any $\mu \in \mathrm{Opt}(n; P_m)$. We will next show that $\beta(\mu; P_m) \leq \frac{20}{m^{m-2}}$ thus completing the proof. Let $x \in [n]$ be such that $\bar{\mu}(x) \neq 0$. By Lemma 2.9 we have

$$\bar{\mu}(x) \cdot (m-1) \cdot \beta(\mu; P_m) \leq 2 \sum_{\substack{P \in \mathbf{C}(P_m, n) \\ V(P) \ni x}} \mu(P) \ . \tag{10}$$

Given distinct $s, t \in [n]$ and $0 \leq \ell \leq m-2$ we define $\mathbf{C}^*(s, t, \ell)$ to be the set of all copies of $P_{(\ell, m-\ell-2)}$ in K_n, where the path of length ℓ starts with s and the path of length $m - \ell - 2$ starts with t. We have

$$\begin{aligned}
\sum_{\substack{P \in \mathbf{C}(P_m, n) \\ V(P) \ni x}} \mu(P) &\leq \sum_{y \in [n] \setminus \{x\}} \mu(x, y) \sum_{\ell=0}^{m-2} \sum_{P \in \mathbf{C}^*(x, y, \ell)} \mu(P) \\
&\leq \sum_{y \in [n] \setminus \{x\}} \mu(x, y) \sum_{\ell=0}^{m-2} \beta^*_{1,n}(P_{(\ell, m-\ell-2)}) \\
&\leq \frac{m-1}{(m-2)^{m-2}} \sum_{y \in [n] \setminus \{x\}} \mu(x, y) = \frac{\bar{\mu}(x)(m-1)}{(m-2)^{m-2}} \ , \qquad (11)
\end{aligned}$$

where the second inequality holds by the definition[4] of $\beta^*_{1,n}(P_{(s,t)})$, and the third inequality holds by Lemma 2.5. Recalling that $\bar{\mu}(x) > 0$ and combining (10) and (11) we infer that

$$\beta(\mu; P_m) \leq \frac{2}{(m-2)^{m-2}} \leq \frac{20}{m^{m-2}} . \qquad \blacksquare$$

3 Proving the main result

We start this section with stating the optimization problem of Cox and Martin [2].

Definition 3.1. *Let n be an integer and let μ be a probability measure on the edges of K_n. Then, for any integer $m \geq 2$, letting $(n)_m$ to be the set of all ordered m-tuples of distinct elements from $[n]$, define*

$$\rho(\mu; m) = \sum_{x \in (n)_m} \bar{\mu}(x_1) \left(\prod_{i=1}^{m-1} \mu(x_i, x_{i+1}) \right) \bar{\mu}(x_m) .$$

Furthermore, define

$$\rho_n(m) = \sup_{\mu} \rho(\mu; m) \quad \text{and} \quad \rho(m) = \sup_{n \in \mathbb{N}} \rho_n(m) ,$$

where the supremum in the definition of $\rho_n(m)$ is taken over all probability measures μ on the edges of K_n.

Note that if we expand the products in the definition of $\rho(m)$ we see that $\rho(m)$ is very similar to $\beta(P_{m+2})$. The crucial difference is that in $\rho(m)$ we count the total weight of walks of a very special structure. These walks are formed by first choosing distinct $x_2, \ldots, x_{m+1}$ to be a copy of P_m, and then choosing *arbitrary* $x_1 \neq x_2$ and $x_{m+2} \neq x_{m+1}$ (so we allow $x_1 = x_{m+1}$ and/or $x_1, x_{m+2} \in \{x_2, \ldots, x_{m+1}\}$). For example, a walk of this type might be $(1, 2, 1, 2)$ or $(1, 2, 1, 3, 1)$.

Our main task in this section is to prove the following lemma.

Lemma 3.2. *For every integers $n \geq m \geq 2$ we have*

$$\rho_n(m) \leq \frac{1000}{m^2} \cdot \beta(P_m) .$$

The constant 1000 in the above lemma is clearly not optimal. We did not make any attempt to improve it, as it seems that a new idea is required to obtain the optimal one. A simple lower bound for $\rho_n(m)$ is $8/m^m$, which is achieved by the uniform distribution on the edges of C_m. As we mentioned in the previous section, it seems reasonable to conjecture that $\beta(P_m) = 1/m^{m-2}$. Therefore, a natural conjecture is that in Lemma 3.2 the optimal constant is 8.

Let us first deduce Theorem 1.1 from Lemmas 2.2 and 3.2.

Proof of Theorem 1.1. Lemma 2.3 in Cox and Martin [2] asserts that for all $m \geq 2$ we have

$$N_{\mathcal{P}}(n, P_{2m+1}) \leq (\rho(m)/2 + o(1)) n^{m+1} .$$

[4] We rely on the fact that although $\beta^*_{w,n}(P_{(s,t)})$ was defined with respect to paths starting at vertices 1 and n, we could have chosen any pair of vertices in $[n]$ (in the above proof we use x, y).

Furthermore, since Lemma 3.2 holds for all n, we deduce that $\rho(m) \leq \frac{10^3}{m^2} \cdot \beta(P_m)$. Together with Lemma 2.2, this gives Theorem 1.1 as then

$$\begin{aligned} N_{\mathcal{P}}(n, P_{2m+1}) &\leq (\rho(m)/2 + o(1))n^{m+1} \leq (500\beta(P_m)m^{-2} + o(1))n^{m+1} \\ &\leq (10^4 m^{-m} + o(1))n^{m+1} . \end{aligned}$$

■

Before proving Lemma 3.2, let us recall a special case of the Karush–Kuhn–Tucker (KKT) conditions (see Corollaries 9.6 and 9.10 in [5]).

Theorem 3.3 (Special case of the KKT conditions)**.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function, and consider the optimization problem*

$$\max_{x \in \Delta} f(x) \ , \ \text{ where } \Delta = \left\{ x : \sum_{i=1}^{n} x_i = 1 \text{ and } x_1, \ldots, x_n \geq 0 \right\} .$$

If $\mathbf{x}^*$ achieves this maximum, then there is some $\lambda \in \mathbb{R}$ such that, for each $i \in [n]$, either

$$\mathbf{x}_i^* = 0, \quad \text{or} \quad \frac{\partial f}{\partial x_i}(\mathbf{x}^*) = \lambda \, .$$

Proof (of Lemma 3.2). Let $\mathbf{P}^*$ be the set of walks $(x_1, x_2, \ldots, x_{m+2})$ on $[n]$ constructed as follows: first, choose $(x_2, x_3, \ldots, x_{m+1})$ to be a path (i.e. a copy of P_m), and then complete the walk by choosing an arbitrary $x_1 \neq x_2$ and an arbitrary $x_{m+2} \neq x_{m+1}$. Further, for any $i \neq j \in [n]$ we let $\mathbf{P}^*(\{i,j\})$ be the set of all walks $(x_1, x_2, \ldots, x_{m+2}) \in \mathbf{P}^*$ such that there is k with $\{x_k, x_{k+1}\} = \{i, j\}$.

Define $f\colon \mathbb{R}^{\binom{[n]}{2}} \to \mathbb{R}$ by

$$f(\mathbf{x}) = \sum_{p \in (n)_m} \left(\sum_{p_0 \in [n] \setminus \{p_1\}} \mathbf{x}_{p_0,p_1} \right) \left(\prod_{i=1}^{m-1} \mathbf{x}_{p_i,p_{i+1}} \right) \left(\sum_{p_{m+1} \in [n] \setminus \{p_m\}} \mathbf{x}_{p_m,p_{m+1}} \right) .$$

Suppose μ is a probability measure on the edges of K_n with $\rho(\mu; m) = \rho_n(m)$. When viewing μ as a vector in $\mathbb{R}^{\binom{[n]}{2}}$, we have $f(\mu) = \rho(\mu; m)$, and moreover,

$$f(\mu) = \max_{\mathbf{x} \in \Delta} f(\mathbf{x}) \ , \ \text{ where } \Delta = \left\{ \mathbf{x} : \sum_{i=1}^{\binom{n}{2}} \mathbf{x}_i = 1 \text{ and } \mathbf{x}_1, \ldots, \mathbf{x}_{\binom{n}{2}} \geq 0 \right\} .$$

By the maximality of μ and by Theorem 3.3 (the KKT conditions), there is a non-negative[5] real λ such that for all $\{i,j\} \in \binom{[n]}{2}$ we have

$$\mu(i,j) = 0 \quad \text{or} \quad \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}_{i,j}}(\mu) = \lambda \, .$$

Note that the degree of each term $\mathbf{x}_{i,j}$, in every monomial of $f(\mathbf{x})$ is at most[6] 3. Thus, for every $\{i,j\} \in \binom{[n]}{2}$ we have

$$\lambda \cdot \mu(i,j) = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}_{i,j}}(\mu) \cdot \mu(i,j) \leq 3 \sum_{P \in \mathbf{P}^*(\{i,j\})} \mu(P) \, . \tag{12}$$

[5] As the polynomial has only positive coefficients, λ must be non-negative.

[6] The only case where it is 3 is when $m = 2$ and we consider a walk on one edge three times, e.g, the walk $(1, 2, 1, 2)$.

We also have the following:

$$\begin{aligned}
\lambda &= \sum_{\{i,j\}\in\binom{[n]}{2}} \lambda\cdot\mu(i,j) && (13)\\
&= \sum_{\{i,j\}\in\binom{[n]}{2}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}_{i,j}}(\mu)\cdot\mu(i,j)\\
&\geq \sum_{\{i,j\}\in\binom{[n]}{2}} \sum_{P\in\mathbf{P}^*(\{i,j\})} \mu(P)\\
&= \sum_{P\in\mathbf{P}^*} \mu(P) \sum_{\{i,j\}\in\binom{[n]}{2}} \mathbb{1}(\{i,j\}\in E(P)) && (14)\\
&\geq (m-1)\rho(\mu;m)\ , && (15)
\end{aligned}$$

where the first equality holds as μ is a probability measure, the second equality holds by the definition of λ, and the last inequality holds as there are at least $m-1$ distinct edges in each walk in $\mathbf{P}^*$. Combining (12) and (15) we have the following for all $i\in[n]$:

$$\begin{aligned}
(m-1)\cdot\bar{\mu}(i)\cdot\rho(\mu;m) &= \sum_{j\in[n]\setminus\{i\}} (m-1)\mu(i,j)\rho(\mu;m)\\
&\leq 3\sum_{j\in[n]\setminus\{i\}} \sum_{P\in\mathbf{P}^*(\{i,j\})} \mu(P)\\
&= 3\sum_{P\in\mathbf{P}^*} \mu(P) \sum_{j\in[n]\setminus\{i\}} \mathbb{1}(\{i,j\}\in E(P))\\
&= 3\sum_{P\in\mathbf{P}^*} \deg_P(i)\mu(P)\\
&\leq 12\sum_{P\in\mathbf{P}^*} \mu(P) = 12\cdot\rho(\mu;m)\ .
\end{aligned}$$

where the last inequality follows as for every $P\in\mathbf{P}^*$ and $i\in P$ we have[7] $\deg_P(i)\leq 4$. Dividing both sides by $(m-1)\cdot\rho(\mu;m)$ we obtain that for all i we have $\bar{\mu}(i)\leq\frac{12}{m-1}$. Therefore, as $m\geq 2$ we have

$$\begin{aligned}
\rho(\mu;m) &= \sum_{x\in(n)_m} \bar{\mu}(x_1)\left(\prod_{i=1}^{m-1}\mu(x_i,x_{i+1})\right)\bar{\mu}(x_m)\\
&\leq \frac{144}{(m-1)^2}\sum_{x\in(n)_m}\prod_{i=1}^{m-1}\mu(x_i,x_{i+1}) \leq \frac{1000}{m^2}\cdot\beta(P_m)\ . \qquad \blacksquare
\end{aligned}$$

References

[1] N. Alon and Y. Caro, On the number of subgraphs of prescribed type of planar graphs with a given number of vertices, Discrete Math. 87 (1984), 25–36. 1

[7] An example being the walk $(1,2,1,3,1)$.

[2] C. Cox and R. R. Martin, Counting paths, cycles, and blow-ups in planar graphs, J. Graph Theory. (2022), 1–38. 1, 1.1, 2, 2, 2.9, 3, 3

[3] D. Eppstein, Connectivity, graph minors, and subgraph multiplicity, J. Graph Theory 17 (1993), 409—416. 1

[4] D. Ghosh, E. Győri, R. R. Martin, A. Paulos, N. Salia, C. Xiao, and O. Zamora, The maximum number of paths of length four in a planar graph, Discrete Math. 344 (2021), article 112317. 1

[5] O. Güler, **Foundations of optimization**, Springer Science, (2010). 3

[6] E. Győri, A. Paulos, N. Salia, C. Tompkins, and O. Zamora, The maximum number of pentagons in a planar graph, arXiv preprint:1909.13532 (2019). 1

[7] E. Győri, A. Paulos, N. Salia, C. Tompkins, and O. Zamora, Generalized planar Turán numbers, Electron. J. Comb. 28 (2021), P4.32. 1

[8] E. Győri, A. Paulos, N. Salia, C. Tompkins, and O. Zamora, The maximum number of paths of length three in a planar graph, arXiv preprint:1909.13539 (2021). 1

[9] S. L. Hakimi and E. E. Schmeichel, On the number of cycles of length k in a maximal planar graph, J. Graph Theory 3 (1979), 69–86. 1

[10] T. Huynh, G. Joret, and D. Wood, Subgraph densities in a surface, Comb. Probab. Comput. (2020), 1—28. 1

[11] Z. Lv, E. Győri, Z. He, N. Salia, C. Tompkins, and X. Zhu, The maximum number of copies of an even cycle in a planar graph, arXiv preprint:2205.15810 (2022). 1.1

[12] N. Wormald, On the frequency of 3-connected subgraphs of planar graphs, Bull. Aust. Math. Soc. 34 (1986), 309—317. 1