Topics in Extremal Combinatorics (0366.4996)- Fall $'21$

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Home Assignment 4

Due date: 04/12/22

Please submit organized and well written solutions!

Problem 1. Suppose $\mathcal{H} = (V, E)$ is a 3-uniform 6-regular hypergraph. Show that it is possible to split E into two sets E_1, E_2 so that both edge sets cover V, that it $\bigcup_{e \in E_1} e = \bigcup_{e \in E_2} e = V$.

Problem 2. For an integer k, let T_k be the complete k-ary tree of depth k. Let $\mathcal{H}_1$ be the set system whose ground set is the edges of T_k and has a set for each path of T_k starting at the root and ending at a leaf. Let $\mathcal{H}_2$ be the set system whose ground set is the edges of T_k and has a set for each internal vertex $v \in T_k$ that consists of the edges connecting v to its children. Show that $\text{disc}(\mathcal{H}_2) \leq 1$, $\text{disc}(\mathcal{H}_2) \leq 1$ and $\text{disc}(\mathcal{H}_1 \cup \mathcal{H}_2) \geq k$.

Problem 3. Let A be an $n \times n$ matrix of ± 1. Show that there is $x \in \{1, -1\}^n$ so that for every $1 \leq i \leq n$, the i^{th} entry of Ax is smaller, in absolute value, than $2i$.

Problem 4. Show that if $S_1, \ldots, S_n$ are subsets of $[r]$, then there is a $\{-1, 0, 1\}$ valued function that induces discrepancy at most $C\sqrt{r \log(2n/r)}$ and attains the value 0 at most $9r/10$ times.

Problem 5. Show that the eigenvalue bound for the discrepancy of a set system gives only a $\Theta(1)$ lower bound for the set system defined by a Hadamard matrix.

Problem 6. Given a hypergraph $\mathcal{H}$ let $\mathcal{H}_t$ be the sub-hypergraph containing only the edges of size at least t, and let $\Delta(\mathcal{H})$ denote $\mathcal{H}$'s maximum degree. We proved in class that if $\mathcal{H}$ is an n-vertex hypergraph with m edges and there is some t, so that $\Delta(\mathcal{H}_t) \leq t$, then $\mathcal{H}$'s discrepancy is $O(\sqrt{t} \cdot \log n \cdot \sqrt{\log m})$. Use this to prove that the discrepancy of arithmetic progressions within $\{1, \ldots, n\}$ is $O(n^{1/4} \log^C n)$.
Hint: Consider only arithmetic progressions whose length is a power of 2 with a special structure.