Topics in Extremal Combinatorics (0366.4996) - Spring '15

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Home Assignment 3

Due date: 11/05/15

Please submit organized and well written solutions!

Problem 1. \( G \) is an \((n,d,\lambda)\)-graph if \( G \) is an \( n \)-vertex \( d \)-regular graph satisfying the following property: if \( S \subseteq V(G) \) is of size \( \alpha n \) then
\[
\left| e(S) - \frac{1}{2} d\alpha^2 n \right| \leq \frac{1}{2} \lambda \alpha (1 - \alpha) n .
\]
Show that there are absolute constants \( \beta, \delta > 0 \) and \( d_0 \) so that the following holds for all \( d \geq d_0 \) and \( n \geq n_0(d) \): if \( G \) is an \((n,d,\delta d)\)-graph then every 2-coloring of \( G \) contains a monochromatic path of length \( \beta n \).

Problem 2. We’ve seen that if every \( U \subset V(G) \), \( |U| \leq u \) satisfies \( |N(U)| \geq 2|U| \), then \( G \) contains \( P_{3u-1} \). Prove that under the same assumption we can actually find a cycle of length at least \( 3u \).

Problem 3. Show that for every \( \epsilon \) there is \( C = C(\epsilon) \) so that if \( G \) has \( \epsilon n^2 \) edges and no independent set of size \( n / 2^{C \sqrt{\log n}} \) then \( G \) contains a \( K_4 \).

Problem 4. Show that if \( H \) is an \( r \)-degenerate bipartite graph then \( ex(n,H) \leq cn^{2 - \frac{1}{8r}} \).

Problem 5. Show that if \( G \) is a bipartite graph with \( m \) edges and no isolated vertices then \( r(G) \leq 2^{O(\sqrt{m})} \).