Home Assignment 4

Due date: 01/06/15

Please submit organized and well written solutions!

Problem 1. Suppose $H = (V, E)$ is a 3-uniform 6-regular hypergraph. Show that it is possible to split $E$ into two sets $E_1, E_2$ so that both edge sets cover $V$, that is $\bigcup_{e \in E_1} e = \bigcup_{e \in E_2} e = V$.

Problem 2. For an integer $k$, let $T_k$ be the complete $k$-ary tree of depth $k$. Let $H_1$ be the set system whose ground set is the edges of $T_k$ and has a set for each path of $T_k$ starting at the root and ending at a leaf. Let $H_2$ be the set system whose ground set is the edges of $T_k$ and has a set for each internal vertex $v \in T_k$ that consists of the edges connecting $v$ to its children. Show that $\text{disc}(H_2) \leq 1$, $\text{disc}(H_2) \leq 1$ and $\text{disc}(H_1 \cup H_2) \geq k$.

Problem 3. Let $A$ be an $n \times n$ matrix of $\pm 1$. Show that there is $x \in \{1, -1\}^n$ so that for every $1 \leq i \leq n$, the $i^{th}$ entry of $Ax$ is smaller, in absolute value, than $2i$.

Problem 4. Show that if $S_1, \ldots, S_n$ are subsets of $[r]$, then there is a $\{-1, 0, 1\}$ valued function that induces discrepancy at most $C \sqrt{r \log(2n/r)}$ and attains the value 0 at most $9r/10$ times.

Problem 5. Show that the eigenvalue bound for the discrepancy of a set system gives only a $\Theta(1)$ lower bound for the set system defined by a Hadamard matrix.

Problem 6. Given a hypergraph $H$ let $H_t$ be the sub-hypergraph containing only the edges of size at least $t$, and let $\Delta(H)$ denote $H$’s maximum degree. We proved in class that if $H$ is an $n$-vertex hypergraph with $m$ edges and there is some $t$, so that $\Delta(H_t) \leq t$, then $H$’s discrepancy is $O(\sqrt{t} \cdot \log n \cdot \sqrt{\log m})$. Use this to prove that the discrepancy of arithmetic progressions within $\{1, \ldots, n\}$ is $O(n^{1/4} \log^{C} n)$.

Hint: Consider only arithmetic progressions whose length is a power of 2 with a special structure.