A COMBINATORIAL CRITERION FOR MACROSCOPIC CIRCLES IN PLANAR TRIANGULATIONS

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Abstract. Given a finite simple triangulation, we estimate the sizes of circles in its circle packing in terms of Cannon's [1] vertex extremal length. Our estimates provide control over the size of the largest circle in the packing. We use them, combined with results from [12], to prove that in a proper circle packing of the discrete *mating-of-trees* random map model of Duplantier, Gwynne, Miller and Sheffield [7, 13], the size of the largest circle goes to zero with high probability.

1. Introduction

Koebe's circle packing theorem [15] (see also [17, 22]) is a canonical and widely used method of drawing planar maps. Various geometric properties of the circle packing encode important probabilistic information of the map. For example, a landmark result of He and Schramm [14] states that a bounded degree one-ended triangulation is recurrent if and only if its circle packing has no accumulation points.

In this paper, we estimate the sizes of circles in the circle packing of a planar triangulation in terms of Cannon's [1] *vertex extremal length*. In particular, we provide an if-and-only-if criterion for the property that all the circles in the packing are small. This property is fundamentally important and is believed to hold in all natural models of random planar maps. Proving it for a random simple triangulation on n vertices is an important open problem (see [16, Section 6]).

We use our criterion together with estimates of [12] to prove that the size of the largest circle in the circle packing of the discrete *mating-of-trees* model of Duplantier, Gwynne, Miller and Sheffield [7, 13] goes to zero with high probability. When combined with the main theorem of [10], this shows that discrete analytic functions on the circle packing embedding of the mating-of-trees map approximate classical analytic functions on the domain of the circle packing. See the discussion in Section 1.3.

1.1. Circle packing and vertex extremal length.

A **circle packing** of a simple connected planar map G with vertex set V is a collection $\mathcal{P} = \{C_v\}_{v \in V}$ of circles in the plane with disjoint interiors such that C_v is tangent to C_u if and only if u and v form an edge. We further require that for each vertex v, the cyclic order of the circles tangent to C_v agrees with the cyclic order of the neighbors of v in G. Koebe's *circle packing theorem* mentioned above asserts that any simple connected planar map has a circle packing. Furthermore, when the map is a triangulation, the packing is unique up to Möbius transformations. See [17] for details.

A **triangulation with boundary** is a finite simple connected planar map in which all faces are triangles except for the outer face whose boundary is a simple cycle. Every triangulation with boundary has a circle packing in which the circles corresponding to vertices of the outer face are internally tangent to the unit circle $\partial\mathbb{D} = \{z : |z| = 1\}$ and all other circles are contained in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ [17, Claim 4.9]. We call this a "circle packing in $\mathbb{D}$." This packing is unique up to Möbius transformations from $\mathbb{D}$ onto itself. A **rooted triangulation with boundary** is a pair (G, ρ) where G is a triangulation with boundary and ρ is a vertex in $V = V(G)$ that does not belong to the outer face. For any such (G, ρ), denote by $\mathcal{P}_\rho = \{C_v\}_{v\in V}$ a circle packing of G in $\mathbb{D}$ so that the circle C_ρ is centered at the origin. By the aforementioned uniqueness, this circle packing is unique up to rotations. Therefore, the radius of each C_v, which we denote by $\mathrm{rad}_\rho(v)$, is well-defined.

Our bounds use the notion of *vertex extremal length*, introduced by Cannon [1]. Let $G = (V, E)$ be a graph. Given a function $m : V \to [0, \infty)$ and a finite path γ in G, we define the **length** of γ according to m as

$$\mathrm{len}_m(\gamma) = \sum_{v\in\gamma} m(v)\,.$$

Given a set of finite paths Γ in G we define

$$\mathrm{len}_m(\Gamma) = \inf_{\gamma\in\Gamma} \mathrm{len}_m(\gamma)\,.$$

The **vertex extremal length** of a set of finite paths Γ in G is now defined to be

$$\mathrm{VEL}(\Gamma) = \sup_{m:V\to[0,\infty)} \frac{\mathrm{len}_m(\Gamma)^2}{\mathrm{area}(m)}\,,$$

where $\mathrm{area}(m) = \sum_{v\in V} m(v)^2$.

Suppose that (G, ρ) is a rooted triangulation with boundary. Given a vertex $v \neq \rho$, we denote by $\Gamma_{\rho,v}$ the set of paths in G starting and ending at v that have winding number 1 around ρ. Also, for any vertex v we denote by Γ_v^∂ the set of paths in G starting at v and ending at a vertex of the outer face. Our first main result is the following.

Theorem 1.1. *Let (G, ρ) be a rooted triangulation with boundary and set $V = V(G)$. For any $v \in V$ distinct from ρ,*

$$\mathrm{VEL}(\Gamma_{\rho,v}) \le 4\,\mathrm{rad}_\rho(v)^{-1}\,. \tag{1}$$

Furthermore, for any $v \in V$,

$$\mathrm{VEL}(\Gamma_v^\partial) \le 1 + \frac{1}{2}\log\left(\frac{1}{\mathrm{rad}_\rho(v)}\right)\,. \tag{2}$$

The next theorem provides a bound in the other direction. For convenience we impose the convention that $\mathrm{VEL}(\Gamma_{\rho,\rho}) = 0$.

Theorem 1.2. *Let (G, ρ) be a rooted triangulation with boundary and set $V = V(G)$. Let $\mathcal{P}_\rho = \{C_v\}_{v\in V}$ be a circle packing of G in $\mathbb{D}$ such that C_ρ is centered at the origin. If all the circles in $\mathcal{P}_\rho$ have radius at most ε, then*

$$\min_{v\in V}\Big(\mathrm{VEL}(\Gamma_{\rho,v}) \vee \mathrm{VEL}(\Gamma_v^\partial)\Big) \ge c\log(1/\varepsilon) \tag{3}$$

for some universal constant $c > 0$.

We say that a sequence (G_n, ρ_n) of rooted triangulations with boundary has **no macroscopic circles** if $\max_{v\in V(G_n)} \mathrm{rad}_{\rho_n}(v) \to 0$ as $n \to \infty$. Combining the two theorems above, we obtain an if-and-only-if criterion for the existence of macroscopic circles.

Corollary 1.3. *Let (G_n, ρ_n) be a sequence of rooted triangulations with boundary. Then (G_n, ρ_n) has no macroscopic circles if and only if*

$$\min_{v\in V(G_n)} \Big(\mathrm{VEL}(\Gamma_{\rho_n,v}) \vee \mathrm{VEL}(\Gamma_v^{\partial})\Big) \underset{n\to\infty}{\longrightarrow} \infty\,.$$

The vertex extremal length is an "embedding-invariant" quantity that encodes various geometric properties of embeddings in which each vertex corresponds to a cell whose squared diameter is comparable to its area. The relationship between squared diameter and area might be precise (as in the case of circle packing) or might hold only in a rough averaged sense. Either way, we can use vertex extremal length to infer that if one such embedding of a map does not possess macroscopic cells, then neither does the other. We carry out this procedure in Sections 1.2 and 2.3 for the "mating-of-trees" random map model, which we now describe.

1.2. **Mating of trees.** The discrete "mating-of-trees" is a random map model constructed and studied by Duplantier, Gwynne, Miller and Sheffield [7, 13, 12]. The model is parametrized by a real number $\gamma \in (0, 2)$ and is constructed to be in the universality class of the γ-LQG surface. Given $\gamma \in (0, 2)$ fixed, for each $\varepsilon > 0$ the model defines an infinite random planar triangulation $\mathcal{G}^\varepsilon$ on the vertex set $\varepsilon\mathbb{Z}$. In this paper we will not use the definition of $\mathcal{G}^\varepsilon$ directly, instead using properties of $\mathcal{G}^\varepsilon$ that have been proved in [7, 13, 12]. Nevertheless, for the sake of completeness we now provide the definition of $\mathcal{G}^\varepsilon$ as it appears in [12].

Start with a standard two-sided planar Brownian motion which is at the origin at time 0. Apply an appropriate linear transformation so that both coordinates of the resulting process $(L, R) : \mathbb{R} \to \mathbb{R}^2$ are standard two-sided linear Brownian motions with correlation $\mathrm{corr}(L_t, R_t) = -\cos(\pi\gamma^2/4)$ for all $t \in \mathbb{R}$ except $t = 0$ (where $L_0 = R_0 = 0$). Given $\varepsilon > 0$, draw the planar map $\mathcal{G}^\varepsilon$ on the vertex set $\varepsilon\mathbb{Z}$ by first drawing an edge between each pair of consecutive vertices $x, x + \varepsilon$ so that the union of all these edges is a horizontal line. Next, for each pair of vertices $x_1, x_2 \in \varepsilon\mathbb{Z}$ such that $x_2 > x_1 + \varepsilon$ and

$$\left(\inf_{t\in[x_1-\varepsilon,x_1]} L_t\right) \vee \left(\inf_{t\in[x_2-\varepsilon,x_2]} L_t\right) \le \inf_{t\in[x_1,x_2-\varepsilon]} L_t\,, \tag{4}$$

draw an edge between x_1 and x_2 in the space below the horizontal line. Finally, for each pair of vertices $x_1, x_2 \in \varepsilon\mathbb{Z}$ such that $x_2 > x_1 + \varepsilon$ and (4) holds with R_t in place of L_t, draw an edge between x_1 and x_2 in the space above the horizontal line. Figure 1 shows a geometric interpretation of this process. The map $\mathcal{G}^\varepsilon$ is almost surely a triangulation, with each pair of vertices connected by at most two edges.

Many natural models of random planar maps can be encoded by a discrete version of the mating-of-trees map in which the Brownian motions L, R are replaced by discrete time random walks. See [12] for references. Thus the mating-of-trees map can be understood as a coarse-grained approximation of these models and can be used to derive new results about them. For instance, Gwynne and Miller [11] have recently shown using this approach that the spectral dimension of the *uniform infinite planar triangulation* (UIPT) is almost surely 2.

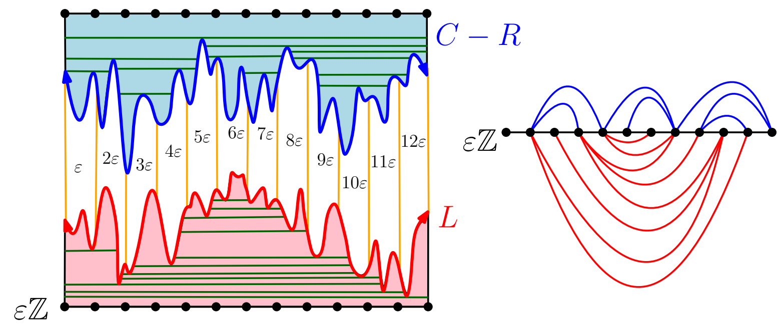


FIGURE 1. To draw the map $\mathcal{G}^\varepsilon$, choose a constant C large enough that the graphs of the Brownian motions L_t and $C - R_t$ do not intersect for t in a given interval. The left image shows the graphs of L and $C - R$ on the interval $[0, 12\varepsilon]$. Each $x \in \varepsilon\mathbb{Z}$ corresponds to the vertical strip $\{(t, y) : t \in [x - \varepsilon, x],\ y \in [L_t, C - R_t]\}$, which is bounded by orange vertical lines. Given $x_1, x_2 \in \varepsilon\mathbb{Z}$ with $x_2 > x_1 + \varepsilon$, equation (4) or its analogue for R_t is satisfied if and only if one can draw a horizontal line between the strips corresponding to x_1 and x_2 that stays below L or above $C - R$, respectively. Such horizontal lines are drawn in green (including some whose endpoints are outside the pictured interval). The right image is a drawing of $\mathcal{G}^\varepsilon$ where the edges coming from green horizontal lines below L are drawn in red and the edges coming from green horizontal lines above $C - R$ are drawn in blue. Images are from [12] and are reproduced with permission.

A remarkable feature of the map $\mathcal{G}^\varepsilon$ is that it comes with an *a priori embedding* in $\mathbb{C}$. See [7, Theorem 1.9] and [13, Proposition 2.2]. This embedding plays a central role in the precise formulation of our small-circles result, Theorem 1.4, and in its proof. It has the following properties. To each vertex v of $\mathcal{G}^\varepsilon$ there is an associated *cell* H_v^ε, which is a compact connected subset of $\mathbb{C}$ such that the interiors of all cells are pairwise disjoint and vertices are adjacent in $\mathcal{G}^\varepsilon$ if and only if their corresponding cells share a non-trivial connected boundary arc. The union of all cells is the whole plane. We remark that $\mathcal{G}^\varepsilon$ has the same distribution for all $\varepsilon > 0$ (by the scale-invariance of Brownian motion) but the a priori embedding is different. For a set $D \subset \mathbb{C}$ we write

$$\mathcal{V}\mathcal{G}^\varepsilon(D) = \{v \in \mathcal{V}\mathcal{G}^\varepsilon : H_v^\varepsilon \cap D \neq \varnothing\},$$

where $\mathcal{V}\mathcal{G}^\varepsilon$ is the vertex set of $\mathcal{G}^\varepsilon$. The set $\mathcal{V}\mathcal{G}^\varepsilon(D)$ is finite if D is bounded.

Let $B(0, \rho)$ denote the disk $\{z : |z| < \rho\}$. In [12] various estimates are proven concerning the sets $\mathcal{V}\mathcal{G}^\varepsilon(B(0, \rho))$ for $\rho < 1$. For convenience, we scale the a priori embedding by a factor of 2 so that the set $\mathcal{V}\mathcal{G}^\varepsilon(\mathbb{D})$ in our normalization is the same as the set $\mathcal{V}\mathcal{G}^\varepsilon(B(0, 1/2))$ in the language of [12].

Under this scaling, let $\mathcal{G}^\varepsilon(\mathbb{D})$ be the submap of $\mathcal{G}^\varepsilon$ whose vertex set is $\mathcal{V}\mathcal{G}^\varepsilon(\mathbb{D})$ and whose edges are the edges of $\mathcal{G}^\varepsilon$ spanned by $\mathcal{V}\mathcal{G}^\varepsilon(\mathbb{D})$. When ε is small, with high probability the maximal cell diameter in $\mathcal{G}^\varepsilon(\mathbb{D})$ is at most ε^q for some constant $q > 0$ [12, Lemma 2.7]. In the construction that follows, we assume that this event occurs.

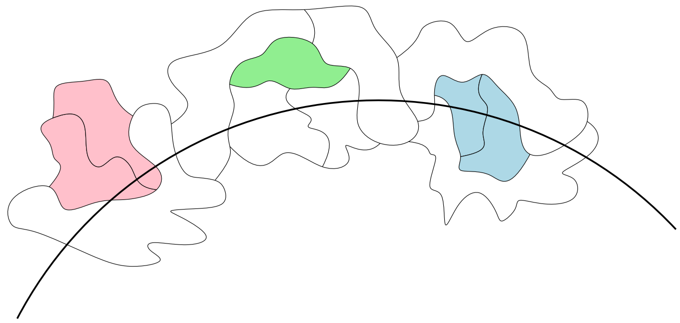


FIGURE 2. Modification of $\mathcal{G}^\varepsilon(\mathbb{D})$ into $\mathcal{G}_2^\varepsilon$. Several cells H_v^ε are shown along with a portion of the unit circle (thick black curve). The transition from $\mathcal{G}^\varepsilon(\mathbb{D})$ to $\mathcal{G}_0^\varepsilon$ deletes the vertices whose cells are colored in red. The transition from $\mathcal{G}_0^\varepsilon$ to $\mathcal{G}_1^\varepsilon$ adds the vertex whose cell is colored in green. Finally, the transition from $\mathcal{G}_1^\varepsilon$ to $\mathcal{G}_2^\varepsilon$ deletes the vertices whose cells are colored in blue and absorbs the blue area either into the cell above or into the cell below.

We make minor modifications to turn $\mathcal{G}^\varepsilon(\mathbb{D})$ into a (simple) triangulation with boundary. These modifications are illustrated in Figure 2. First, if $\mathcal{G}^\varepsilon(\mathbb{D})$ is not already 2-connected, we observe that all the vertices v for which H_v^ε intersects the slightly smaller disk $\{z : |z| < 1 - \varepsilon^q\}$ are contained in the same block (maximal 2-connected component) of $\mathcal{G}^\varepsilon(\mathbb{D})$. We restrict to this block, which we call $\mathcal{G}_0^\varepsilon$. This cuts off the "dangling ends" associated with repeated vertices in the boundary of the outer face of $\mathcal{G}^\varepsilon(\mathbb{D})$; the boundary of the new outer face is a simple cycle [4, Proposition 4.2.5] whose vertices and edges are all part of the old boundary.

The map $\mathcal{G}_0^\varepsilon$ might have inner faces which are not triangles because it is possible for a cycle in $\mathcal{G}_0^\varepsilon$ to enclose vertices in $\mathcal{V}\mathcal{G}^\varepsilon \setminus \mathcal{V}\mathcal{G}_0^\varepsilon$. In this case we simply add to the map all of these enclosed vertices and their incident edges and denote the resulting map by $\mathcal{G}_1^\varepsilon$. The boundary of the outer face is unchanged by this procedure.

Now the inner faces of $\mathcal{G}_1^\varepsilon$ are all triangles, but the map is not necessarily simple. As described above, between any given pair of vertices there might be two edges (but no more than two). At each occurrence of such parallel edges we collapse them to a single edge and erase all the vertices and edges between them. Furthermore, in the embedding we assign the space of the erased cells arbitrarily to one of the two surrounding cells corresponding to the vertices of the double edge. This guarantees simplicity. We denote this map by $\mathcal{G}_2^\varepsilon$ and observe that it is a triangulation with boundary according to the definition in Section 1.1. Indeed, $\mathcal{G}_2^\varepsilon$ is a simple submap of $\mathcal{G}_1^\varepsilon$ with the same outer face. Lastly, the vertex whose corresponding cell contains the origin (or any one of the vertices with this property, in case the origin is on a cell boundary) is declared to be the root and denoted by ρ. Since ρ is not a vertex of the outer face, we have that $(\mathcal{G}_2^\varepsilon, \rho)$ is a rooted triangulation with boundary.

As in Section 1.1, let $\mathcal{P}_\rho^\varepsilon = \{C_v\}_{v \in \mathcal{V}\mathcal{G}_2^\varepsilon}$ be a circle packing of $\mathcal{G}_2^\varepsilon$ in $\mathbb{D}$ such that C_ρ is centered at the origin. This packing is unique up to rotations. Denote the radius of each C_v by $\mathrm{rad}_\rho^\varepsilon(v)$. The following theorem bounds the maximum radius in $\mathcal{P}_\rho^\varepsilon$ with high probability. It implies that for a sequence ε_n tending to 0 sufficiently fast, the sequence of random rooted triangulations with boundary $(\mathcal{G}_2^{\varepsilon_n}, \rho_n)$ almost surely has no macroscopic circles.

Theorem 1.4. *There exist constants $C, c > 0$, depending only on the parameter $\gamma \in (0, 2)$, such that with probability at least $1 - \varepsilon^c$,*

$$\max_{v \in \mathcal{V}\mathcal{G}_2^{\varepsilon}} \operatorname{rad}_{\rho}^{\varepsilon}(v) \le \frac{C}{\log(1/\varepsilon)} .$$

Note that the low probability ε^c encompasses both the event that the modification of $\mathcal{G}^{\varepsilon}(\mathbb{D})$ into $\mathcal{G}_2^{\varepsilon}$ fails due to the presence of a very large cell, and the event that the modification succeeds but the circle packing has a very large circle.

1.3. Discrete complex analysis on random planar maps.

Discrete complex analysis has been pivotal in the study of statistical physics on two-dimensional lattices [19, 20, 2, 3, 21]. Recently, statistical physics on random planar maps has drawn a great deal of attention due to the conjectured KPZ correspondence, see [9, 8]. Random planar maps are combinatorial objects and do not come equipped with a canonical embedding in the plane. Thus, a major challenge is to find an embedding on which discrete complex analysis can be performed. The motivation of the current paper and its companion [10] is to address this challenge. Indeed, the combination of Theorem 1.4 and [10, Theorem 1.1] enables one to perform discrete complex analysis on the circle packing embedding of the mating-of-trees random map model. We provide a brief explanation here and refer the reader to [10] for further details.

A natural candidate for the embedding of a generic planar map is the *orthodiagonal* representation. An *orthodiagonal map* is a plane graph having quadrilateral faces with orthogonal diagonals. It turns out that any simple 3-connected finite planar map, in particular any simple triangulation, can be represented by an orthodiagonal map via the circle packing theorem, see [10, Section 2]. Furthermore, Duffin [6] showed that orthodiagonal maps admit a very natural form of discrete analyticity: a complex-valued function f is said to be **discrete analytic** if for every inner quadrilateral $[v_1, w_1, v_2, w_2]$ of the map,

$$\frac{f(v_2) - f(v_1)}{v_2 - v_1} = \frac{f(w_2) - f(w_1)}{w_2 - w_1} .$$

With this definition it can be shown that discrete contour integrals of discrete analytic functions vanish, and that the real part of a discrete analytic function is discrete harmonic with respect to positive real edge weights induced by the map, mirroring classical complex analysis theory. It is a natural and highly applicable question to ask whether such functions are close to continuous holomorphic functions—the answer is positive.

Indeed, following the work of [5, 18, 23], in [10] we prove a general result regarding the convergence of discrete harmonic functions on orthodiagonal maps to their continuous counterparts.[1] Our main contribution is to drop several local and global regularity conditions, such as bounded vertex degrees, present in the previous results [5, 18, 23]. These conditions have prevented such convergence results from applying to random planar maps. By contrast, our convergence statement

[1] We emphasize that the discrete harmonic functions in [10], just as in [5, 18, 23], are with respect to natural edge weights determined by the orthodiagonal map and not with respect to unit weights. See equation (1) in [10, Section 1.1].

[10, Theorem 1.1] holds for any random map model that has an orthodiagonal representation with maximal edge length going to 0. Theorem 1.4 shows that the circle packing in the unit disk $\mathbb{D}$ of the mating-of-trees random triangulation $\mathcal{G}_2^\varepsilon$ induces such an orthodiagonal representation. For further details, see [10, Corollary 2.2]. It follows that discrete harmonic and analytic functions on the circle packing embedding of $\mathcal{G}_2^\varepsilon$ in $\mathbb{D}$ converge to their continuous counterparts as ε tends to 0.

2. Proofs

2.1. **Proof of Theorem 1.1.** We begin by proving (1). By applying a rotation we may assume that the center of the circle C_v lies on the positive x-axis, and we denote by $t_2 > t_1 > 0$ the two intersection points of C_v with the x-axis. We have that $t_2 - t_1 = 2\mathrm{rad}_\rho(v)$. For each $t \in [t_1, t_2]$ let C_t be the circle of radius t around the origin (this is *not* an element of the circle packing $\mathcal{P}_\rho$). Let γ_t denote the finite simple path in G obtained from C_t by starting at the point $(t, 0)$ and the vertex v, then traversing C_t counterclockwise concatenating to γ_t the vertex of G corresponding to any new circle of $\mathcal{P}_\rho$ that we encounter, until we visit C_v in the last step. See Figure 3a. Note that for almost every $t \in [t_1, t_2]$ the circle C_t does not intersect any tangency point of $\mathcal{P}_\rho$, so γ_t is well defined for this set of t's. Since we start and end at v and wind around the origin a single time, we have that $\gamma_t \in \Gamma_{\rho,v}$ for almost every $t \in [t_1, t_2]$.

Now, given $m : V \to [0, \infty)$ we have that

$$\int_{t_1}^{t_2} \mathrm{len}_m(\gamma_t)dt = \sum_{u \in V} m(u) \int_{t_1}^{t_2} \mathbf{1}_{\{u \in \gamma_t\}} dt\,.$$

The vertices u that contribute to the sum are those for which the interior of C_u intersects the annulus $A = \{z : t_1 \le |z| \le t_2\}$. Given such a circle C_u, label its closest point to the origin by $r_1 e^{i\varphi}$ and its farthest point from the origin by $r_2 e^{i\varphi}$ (the angles are the same). Then

$$\int_{t_1}^{t_2} \mathbf{1}_{\{u \in \gamma_t\}} dt = (r_2 \wedge t_2) - (r_1 \vee t_1) = 2\mathrm{rad}'_\rho(u)$$

where $\mathrm{rad}'_\rho(u)$ is the radius of the circle C'_u for which the line segment between $(r_1 \vee t_1)e^{i\varphi}$ and $(r_2 \wedge t_2)e^{i\varphi}$ is a diameter. One such circle C'_u is illustrated in Figure 3a. The circles C'_u are all inside the annulus A, and they are internally disjoint because each C'_u is contained inside C_u. Hence

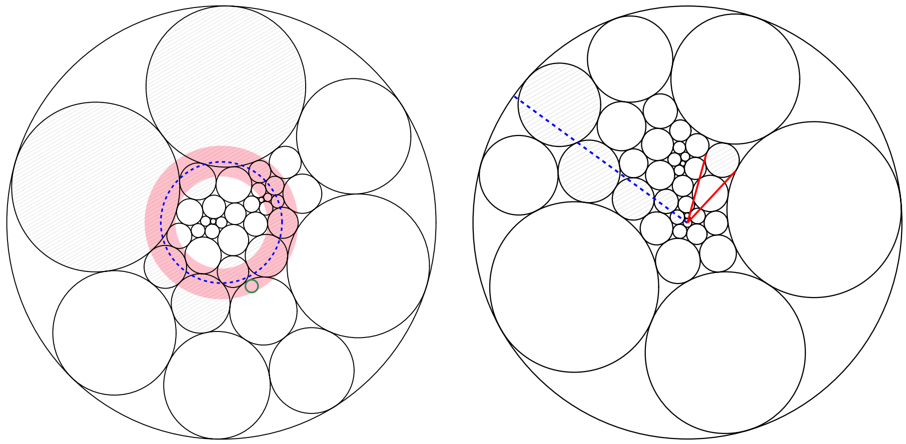
$$\sum_u (\mathrm{rad}'_\rho(u))^2 \le t_2^2 - t_1^2 \le 2(t_2 - t_1)$$

and, by Cauchy-Schwarz,

$$\begin{aligned}\int_{t_1}^{t_2} \mathrm{len}_m(\gamma_t)dt = 2\sum_u m(u)\mathrm{rad}'_\rho(u) &\le 2\left(\mathrm{area}(m)\sum_u (\mathrm{rad}'_\rho(u))^2\right)^{1/2} \\ &\le 2\Big(2\,\mathrm{area}(m)(t_2 - t_1)\Big)^{1/2}.\end{aligned}$$

We deduce that there must exist $t \in (t_1, t_2)$ such that

$$\mathrm{len}_m(\gamma_t) \le \frac{2\sqrt{2\,\mathrm{area}(m)}}{\sqrt{t_2 - t_1}} = 2\sqrt{\frac{\mathrm{area}(m)}{\mathrm{rad}_\rho(v)}}\,.$$



(A) Construction of the path γ_t. The red annulus is $A = \{z : t_1 \leq |z| \leq t_2\}$ and the blue dashed circle is C_t. The circles corresponding to vertices in γ_t are shaded in gray. Additionally, one of the circles C'_u is drawn in green.

(B) Construction of the path γ_θ. The blue dashed line is ℓ_θ and the circles corresponding to vertices in γ_θ are shaded in gray. The solid red segments are the tangent lines from the origin to a circle C_u which is shaded in red.

FIGURE 3. Illustrations of various parts of the proof of Theoeem 1.1.

Plugging this into the definition of $\text{VEL}(\Gamma_{\rho,v})$ concludes the proof of (1).

We now prove (2). If v is a vertex of the outer face of G then $\text{VEL}(\Gamma^\partial_v) = 1$. In all other cases we will show that

$$\text{VEL}(\Gamma^\partial_v) \leq 1 + \frac{1}{2}\log\left(\frac{1}{\text{rad}_v(v)}\right). \quad (5)$$

This implies (2) because $\text{rad}_\rho(v) < \text{rad}_v(v)$ whenever $\rho \neq v$, as we now explain. To get from the packing $\mathcal{P}_v$ to $\mathcal{P}_\rho$, one applies a Möbius transformation of the form $z \mapsto e^{i\theta}(z+w)/(1+\overline{w}z)$ for some $0 < |w| < 1$. This transforms a circle of radius $r < 1$ centered at the origin into a circle of radius $r(1-|w|^2)/(1-|w|^2r^2) < r$.

We consider the packing $\mathcal{P}_v = \{C_u\}_{u\in V}$, where C_v is centered at the origin. For $\theta \in (0, 2\pi)$ let ℓ_θ be the straight line in the plane of angle θ starting from the origin and ending at $\partial\mathbb{D}$. Let γ_θ be the path in G obtained by taking the circles of $\mathcal{P}_v$ that ℓ_θ intersects according to the order in which they are intersected. See Figure 3b. For almost every θ this is a finite simple path in G which starts at v and ends in one of the vertices of the outer face. As before, given any $m : V \to [0,\infty)$ we have that

$$\int_0^{2\pi} \text{len}_m(\gamma_\theta)d\theta = \sum_u m(u)\int_0^{2\pi} \mathbf{1}_{\{u\in\gamma_\theta\}}d\theta\,.$$

For any $u \neq v$, the Lebesgue measure of θ's for which $u \in \gamma_\theta$ is precisely the angle between the two tangents to C_u emanating from the origin, as shown in Figure 3b. If α is half that angle, then $\sin(\alpha) = \text{rad}_v(u)/d(u,v)$ where $d(u,v)$ is the Euclidean distance between the center of C_u and the origin. It is clear that $\alpha \in (0, \pi/2)$ so that $\alpha \le \frac{\pi}{2}\sin(\alpha)$, and we obtain

$$\int_0^{2\pi} \mathrm{len}_m(\gamma_\theta)d\theta \le 2\pi m(v) + \pi \sum_{u\neq v} \frac{m(u)\mathrm{rad}_v(u)}{d(u,v)}$$
$$\le 2\pi\sqrt{\mathrm{area}(m)}\left(1+\frac{1}{4}\sum_{u\neq v}\frac{(\mathrm{rad}_v(u))^2}{d(u,v)^2}\right)^{1/2}$$

where the last inequality is Cauchy-Schwarz.

To bound the sum, let D_u denote the interior of C_u. The function $f(x,y) = 1/(x^2+y^2)$ on $\mathbb{R}^2 \setminus \{0\}$ is subharmonic (this can be checked by computing the Laplacian) and so its value at the center of C_u is less than its average value on D_u. In other words,

$$\frac{1}{d(u,v)^2} \le \frac{1}{\pi\cdot(\mathrm{rad}_v(u))^2}\iint_{D_u}\frac{1}{x^2+y^2}\,dx\,dy\,.$$

The disks D_u for $u \neq v$ are disjoint and all contained in the annulus $A' = \{z : \mathrm{rad}_v(v) \le |z| \le 1\}$. Hence

$$\sum_{u\neq v}\frac{(\mathrm{rad}_v(u))^2}{d(u,v)^2} \le \sum_{u\neq v}\frac{1}{\pi}\iint_{D_u}\frac{1}{x^2+y^2}\,dx\,dy \le \frac{1}{\pi}\iint_{A'}\frac{1}{x^2+y^2}\,dx\,dy$$
$$= 2\int_{\mathrm{rad}_v(v)}^1 \frac{1}{r}\,dr = 2\log\left(\frac{1}{\mathrm{rad}_v(v)}\right)\,.$$

We conclude that

$$\frac{1}{2\pi}\int_0^{2\pi}\mathrm{len}_m(\gamma_\theta)d\theta \le \sqrt{\mathrm{area}(m)}\sqrt{1+\frac{1}{2}\log\left(\frac{1}{\mathrm{rad}_v(v)}\right)}$$

so there is $\theta \in (0,2\pi)$ such that $\mathrm{len}_m(\gamma_\theta)$ has the same bound, proving (5). □

2.2. **Proof of Theorem 1.2.** Because all the paths in $\Gamma_{\rho,v}$ and Γ_v^∂ pass through v, the left side of (3) is at least 1 and we may assume that ε is sufficiently small.

For $z \in \mathbb{D}$ and $r_2 > r_1 > 0$, we denote by $A_z(r_1,r_2)$ the annulus $\{w : r_1 < |z-w| < r_2\}$. Set $K = \lfloor \log_2(1/5\varepsilon)\rfloor - 2$. For $0 \le k \le K$, let $r_k = 2^k(5\varepsilon)$, so that $r_0 = 5\varepsilon$ and $r_K \in [1/8, 1/4]$. Given $v \in V$, let c_v be the center of the circle $C_v \in \mathcal{P}_\rho$ and let B_k be the set of vertices u for which $C_u \subset A_{c_v}(r_{k-1}, r_k)$. We define $m : V \to [0,\infty)$ by

$$m(u) = \begin{cases} \mathrm{rad}_\rho(u)/r_k & \text{if } u \in B_k \text{ for some } 1 \le k \le K \\ 0 & \text{otherwise.}\end{cases}$$

For each $1 \le k \le K$ we have

$$\sum_{u\in B_k} m(u)^2 = \frac{1}{\pi r_k^2}\sum_{u\in B_k}\pi(\mathrm{rad}_\rho(u))^2 \le 1\,,$$

hence $\mathrm{area}(m) \le K$.

If $|c_v| \ge 1/2$ then every path $\gamma \in \Gamma_{\rho,v}$ must visit a vertex v' with $|c_{v'} - c_v| \ge 1/4 \ge r_K$. If $|c_v| \le 1/2$ then the same property holds for every path $\gamma \in \Gamma_v^\partial$.

Suppose γ is a path in G that starts at v and visits a vertex v' such that $|c_{v'} - c_v| \ge r_K$. The packing $\mathcal{P}_\rho$ induces an embedding of G in the plane where each vertex u is drawn at c_u and the edges (u,u') are drawn as the straight lines $c_u c_{u'}$. Under this embedding, the path γ becomes a piecewise linear curve in $\mathbb{D}$ that crosses from the inside to the outside of each annulus $A_{c_v}(r_{k-1}, r_k)$. We abuse notation by referring to this curve also as γ. If γ goes through a vertex $u \in B_k$, then the Euclidean length of $\gamma \cap C_u$ is $2\mathrm{rad}_\rho(u)$ while the contribution of u to $\mathrm{len}_m(\gamma)$ is $\mathrm{rad}_\rho(u)/r_k$, thus the ratio between them is $1/2r_k$. As all the circles in $\mathcal{P}_\rho$ have radius at most ε, any circle C_u whose interior intersects $A_{c_v}(r_{k-1}+2\varepsilon, r_k - 2\varepsilon)$ must be contained entirely in $A_{c_v}(r_{k-1}, r_k)$. It follows that the contribution of B_k to $\mathrm{len}_m(\gamma)$ is at least

$$[(r_k - 2\varepsilon) - (r_{k-1} + 2\varepsilon)]/2r_k \geq 1/20\,.$$

Therefore $\mathrm{len}_m(\gamma) \geq K/20$ and so $(\mathrm{len}_m(\gamma))^2/\,\mathrm{area}(m) \geq K/400$.

Hence, we have that either $\mathrm{VEL}(\Gamma_{\rho,v}) \geq K/400$ or $\mathrm{VEL}(\Gamma_v^\partial) \geq K/400$. □

2.3. **Proof of Theorem 1.4.** We will adapt the argument used to prove Theorem 1.2. In that argument, the smallness of all the circles led to a lower bound on vertex extremal length. In the present setting, results from [12] show that with high probability in the a priori embedding of the mating-of-trees map, all of the cells are small and the sum of the squared diameters of the cells in a sufficiently nice region is proportional to the area of the region. We use this information to repeat the argument from Section 2.2 and get a very similar VEL lower bound, which plugs into Theorem 1.1 to complete the proof.

From the definition of the a priori embedding, it is immediate that given any path $(v_0, v_1, \ldots, v_k)$ in $\mathcal{G}^\varepsilon$ there is a continuous curve in the plane that passes in order through the cells $H^\varepsilon_{v_j}$. The following lemma shows that one can also translate line segments in the plane into paths in $\mathcal{G}^\varepsilon$.

Lemma 2.1. *The a priori embedding $\{H_v^\varepsilon\}_{v\in\mathcal{VG}^\varepsilon}$ almost surely has the following property. Given any line segment z_1z_2 in the plane, let $v_1, v_2 \in \mathcal{VG}^\varepsilon$ satisfy $z_1 \in H^\varepsilon_{v_1}$ and $z_2 \in H^\varepsilon_{v_2}$. Then there is a path in $\mathcal{G}^\varepsilon$ from v_1 to v_2 whose vertices are all in the set $\mathcal{VG}^\varepsilon(z_1z_2)$.*

Proof. This statement is proved in [13] (see the section "Connectivity along lines" in the proof of Proposition 3.1) for horizontal and vertical lines. The proof extends without change to the case of lines at any angle. □

We now quote two results from [12]. For each $v \in \mathcal{VG}^\varepsilon$ we write $\mathrm{diam}(H_v^\varepsilon)$ for the diameter of the cell H_v^ε associated with v, as described in Section 1.2.

Theorem 2.2. *There exist constants $c, q > 0$ (depending only on the parameter $\gamma \in (0,2)$ discussed in Section 1.2) such that for ε sufficiently small,*

$$\mathbb{P}\left(\mathrm{diam}(H_v^\varepsilon) \leq \varepsilon^q \quad \forall\, v \in \mathcal{VG}_1^\varepsilon\right) \geq 1 - \varepsilon^c\,.$$

Proof. Lemma 2.7 of [12] gives the same estimate for all $v \in \mathcal{VG}^\varepsilon(B(0,r))$ for some fixed $r > 1$ (recalling the renormalization by a factor of 2 in Section 1.2). On that event, every $v \in \mathcal{VG}_0^\varepsilon$ has $\mathrm{diam}(H_v^\varepsilon) \leq \varepsilon^q$ and $H_v^\varepsilon \subset B(0, 1+\varepsilon^q)$. If $w \in \mathcal{VG}_1^\varepsilon \setminus \mathcal{VG}_0^\varepsilon$, then w is inside a cycle in $\mathcal{G}^\varepsilon$ of vertices in $\mathcal{VG}_0^\varepsilon$, thus the connected component of w in the subgraph of $\mathcal{G}^\varepsilon$ induced by $\mathcal{VG}^\varepsilon \setminus \mathcal{VG}_0^\varepsilon$ is finite. On the other hand, by Lemma 2.1, a single connected component of this subgraph contains all the vertices $u \in \mathcal{VG}^\varepsilon$ such that H_u^ε intersects the complement of $B(0, 1+\varepsilon^q)$. Indeed, any two such cells $H_u^\varepsilon, H_{u'}^\varepsilon$ can be joined by a piecewise linear curve that avoids the disk $B(0, 1+\varepsilon^q)$. It follows that $H_w^\varepsilon \subset B(0, 1+\varepsilon^q) \subset B(0,r)$ and so the diameter estimate also holds for H_w^ε. □

To state the second result from [12], we recall the notation $A_z(r_1, r_2)$ from Section 2.2. For any open set D we write $\mathrm{area}(D)$ for its Lebesgue measure. Let $q > 0$ be the constant from Theorem 2.2 and set $\mathbb{D}' = \{z : |z| \le 1 + \varepsilon^q\}$. Note that on the event of Theorem 2.2, every $v \in \mathcal{VG}_1^\varepsilon$ has $H_v^\varepsilon \subset \mathbb{D}'$.

Theorem 2.3. *There exist constants $A, c > 0$ (depending only on γ) such that when ε is sufficiently small, for any $|z| \le 1 - \varepsilon^c$ and r_1, r_2 satisfying $1 \ge r_2 \ge r_1 + \varepsilon^c$ we have that*

$$\mathbb{P}\Big(\sum_{v \in \mathcal{VG}^\varepsilon \,:\, H_v^\varepsilon \subset A_z(r_1,r_2) \cap \mathbb{D}'} \mathrm{diam}^2(H_v^\varepsilon) \le A \cdot \mathrm{area}\left(A_z(r_1, r_2) \cap \mathbb{D}'\right)\Big) \ge 1 - \varepsilon^c \,.$$

Proof. Follows directly from Proposition 2.9 of [12] by taking $D = A_z(r_1, r_2) \cap \mathbb{D}'$, noting that $\mathrm{area}(D)$ is at least of order ε^c under our assumptions on z, r_1, r_2. ☐

We now begin the proof of Theorem 1.4. We will show that when ε is sufficiently small, with probability at least $1 - \varepsilon^c$ the map $\mathcal{G}_1^\varepsilon$ satisfies

$$\mathrm{VEL}(\Gamma_{\rho,v}) \ge c \log(1/\varepsilon) \quad \text{for all} \quad v \in \mathcal{VG}_1^\varepsilon \cap \mathcal{VG}^\varepsilon(\{|z| \ge 1/2\}) \tag{6}$$

$$\mathrm{VEL}(\Gamma_v^\partial) \ge c \log(1/\varepsilon) \quad \text{for all} \quad v \in \mathcal{VG}^\varepsilon(\{|z| \le 1/2\})\,, \tag{7}$$

for some fixed constant $c > 0$. By our construction in Section 1.2 we have that $\mathcal{G}_2^\varepsilon$ is a submap of $\mathcal{G}_1^\varepsilon$, hence (6) for $\mathcal{G}_1^\varepsilon$ implies the same bound for $\mathcal{G}_2^\varepsilon$. Furthermore, since both maps have the same outer face, (7) for $\mathcal{G}_1^\varepsilon$ again implies the same estimate for $\mathcal{G}_2^\varepsilon$. Thus, by Theorem 1.1, these two assertions conclude the proof and it suffices to prove (6) and (7) for $\mathcal{G}_1^\varepsilon$.

In fact, we prove only (6); the proof of (7) is very similar. Let $A, c, q > 0$ be the constants from Theorems 2.2 and 2.3. By Theorem 2.2, with probability at least $1 - \varepsilon^c$ we have that $\mathrm{diam}(H_v^\varepsilon) \le \varepsilon^q$ for all $v \in \mathcal{VG}_1^\varepsilon$. Let us assume this event holds.

Set $c' = \min(c, q)/4$ and $K = \lfloor c' \log_2(1/\varepsilon) \rfloor - 3$. For $0 \le k \le K$, set $r_k = 2^{k+1}\varepsilon^{c'}$. Note that $r_0 = 2\varepsilon^{c'}$ and that $r_K \in [1/8, 1/4]$. We would like to draw concentric circles of radius $r_0, \ldots, r_K$ around each cell H_v^ε, as in the proof of Theorem 1.2, but since we will need to use Theorem 2.3 in each annulus we must take care that the total number of annuli is less than ε^{-c}. For this reason, we specify the center points of the circles using the following procedure.

Denote $\mathbb{D}_\varepsilon = \{z : |z| \le 1 - \varepsilon^c\}$. We find points $z_1, \ldots, z_L$ in $\mathbb{D}_\varepsilon$ such that

(1) Every cell corresponding to a vertex $v \in \mathcal{VG}_1^\varepsilon$ is contained in $B(z_i, r_0)$ for some $1 \le i \le L$, and
(2) $L \le 16\varepsilon^{-c/2}$.

Indeed, we take $\{z_1, \ldots, z_L\}$ to be an $\varepsilon^{c'}/2$-net in $\mathbb{D}_\varepsilon$, that is, for any $z \in \mathbb{D}_\varepsilon$ there is some $1 \le i \le L$ such that $|z - z_i| \le \varepsilon^{c'}/2$ and $B(z_i, \varepsilon^{c'}/4) \cap B(z_j, \varepsilon^{c'}/4) = \varnothing$ for any $i \ne j$. Such a set of points can be easily obtained greedily: at each stage, if there is a $z \in \mathbb{D}_\varepsilon$ such that $B(z, \varepsilon^{c'}/4) \cap B(z_i, \varepsilon^{c'}/4) = \varnothing$ for all i, then we add z to the current set of z_i's. Since the added disks are disjoint, this process must end after at most $16\varepsilon^{-2c'}$ steps, so by our choice of c' we have that $L \le 16\varepsilon^{-c/2}$. Due to the maximality of this set of points, we obtain that $\mathbb{D}_\varepsilon \subset \cup_{1 \le i \le L} B(z_i, \varepsilon^{c'}/2)$ and since $c' < \min(c, q)$ we have that $\mathbb{D}' \subset \cup_{1 \le i \le L} B(z_i, \varepsilon^{c'})$. Lastly, for all $v \in \mathcal{VG}_1^\varepsilon$, since $\mathrm{diam}(H_v^\varepsilon) \le \varepsilon^q$ we can find i such that $H_v^\varepsilon \subset B(z_i, 2\varepsilon^{c'})$. We conclude that both (1) and (2) hold for the point set $\{z_1, \ldots, z_L\}$.

We now apply Theorem 2.3 LK times and obtain that with probability at least $1 - LK\varepsilon^c$,

$$\sum_{v \in \mathcal{VG}^\varepsilon \,:\, H_v^\varepsilon \subset A_{z_i}(r_{k-1}, r_k) \cap \mathbb{D}'} \operatorname{diam}^2(H_v^\varepsilon) \le A \cdot \pi r_k^2 \qquad 1 \le i \le L,\ 1 \le k \le K\,. \tag{8}$$

Assume that (8) holds and let $v \in \mathcal{VG}_1^\varepsilon \cap \mathcal{VG}^\varepsilon(\{|z| \ge 1/2\})$. Then $H_v^\varepsilon \subset B(z_i, r_0)$ for some $1 \le i \le L$ by the construction above. Let B_k be the set of vertices $u \in \mathcal{VG}_1^\varepsilon$ for which $H_u^\varepsilon \subset A_{z_i}(r_{k-1}, r_k)$. We define $m : \mathcal{VG}_1^\varepsilon \to [0, \infty)$ by

$$m(u) = \begin{cases} \operatorname{diam}(H_u^\varepsilon)/r_k & \text{if } u \in B_k \text{ for some } 1 \le k \le K \\ 0 & \text{otherwise.} \end{cases}$$

Due to (8), the contribution to $\operatorname{area}(m)$ from each B_k is at most πA, hence $\operatorname{area}(m) \le \pi A K = O(\log(1/\varepsilon))$.

Let γ be any path in $\mathcal{G}_1^\varepsilon$ from v to v that has winding number 1 around ρ. If we write $\gamma = (v_0, v_1, \dots, v_m)$ with $v_0 = v_m = v$, then we may draw a closed curve in the plane that starts and ends at a point $w \in H_v^\varepsilon$ and passes in order through the cells $H_{v_j}^\varepsilon$. We now argue that this curve must cross each of the annuli $A_{z_i}(r_{k-1}, r_k)$ for $1 \le k \le K$. The cell H_ρ^ε either contains the origin or is within distance $2\varepsilon^q$ of the origin in case the cell containing the origin was collapsed during the transition from $\mathcal{G}_1^\varepsilon$ to $\mathcal{G}_2^\varepsilon$. Thus we may choose $z \in H_\rho^\varepsilon$ with $|z| \le 2\varepsilon^q$. Let $w' = -2w/|w|$ be on the opposite side of the origin from w with $|w'| = 2$. We use Lemma 2.1 to convert the straight line zw' into a path in $\mathcal{G}^\varepsilon$ from ρ to $\mathcal{VG}^\varepsilon \setminus \mathcal{VG}_1^\varepsilon$. Since γ has winding number 1 around ρ, it separates ρ from $\mathcal{VG}^\varepsilon \setminus \mathcal{VG}_1^\varepsilon$. (Here we used the property that every vertex in $\mathcal{VG}^\varepsilon$ which is enclosed by a cycle in $\mathcal{G}_1^\varepsilon$ must be an element of $\mathcal{VG}_1^\varepsilon$.) Therefore, this new path intersects γ at some v_j and it follows that the line zw' passes through $H_{v_j}^\varepsilon$. Since $|w| \ge 1/2 - \varepsilon^q$ and $r_K \le 1/4$, in order to visit the cell $H_{v_j}^\varepsilon$ the curve induced by γ must cross all of the annuli.

By our choice of m, for each k the vertices $u \in B_k$ contribute at least $1/2 - O(\varepsilon^{q-c'}) \ge 1/4$ to $\operatorname{len}_m(\gamma)$. Hence $\operatorname{len}_m(\gamma) = \Omega(\log(1/\varepsilon))$ and we conclude that $\operatorname{VEL}(\Gamma_{\rho,v}) = \Omega(\log(1/\varepsilon))$. This confirms (6). The proof of (7) is very similar; we omit the details, mentioning only that if $u \in \mathcal{VG}_1^\varepsilon$ is a vertex of the outer face then H_u^ε must intersect $\partial\mathbb{D}$. ☐

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