Brownian motion — exercise sheet 2

Due on Wednesday, May 12th.

Notation:

- $\{B(t) : t \in [0, \infty)\}$ is standard Brownian motion started at 0.
- For $a \in \mathbb{R}$ we denote $T_a = \inf\{t \geq 0 : B(t) = a\}$.

1. (a) Show that if $u(t, x)$ is a polynomial in t and x such that

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} = 0\,,$$

then $u(t, B(t))$ is a martingale with respect to the standard filtration.

   (b) Conclude that $B(t)^3 - 3tB(t)$ and $B(t)^4 - 6tB(t)^2 + 3t^2$ are martingales.

2. Let $a < 0 < b$ and set $T = \min\{T_a, T_b\}$. Compute $\mathbb{E}T^2$.

3. Show that there exists constants $C, c > 0$ such that for Brownian motion started at 0 and all $n \geq 1$

$$\frac{c}{\sqrt{n}} \leq \mathbf{P}(T_{-1} \geq n) \leq \frac{C}{\sqrt{n}}\,.$$

4. Let $\{X_k\}$ be a sequence of i.i.d. random variables with mean 0 and variance 1 and put $S_n = \sum_{k=1}^{n} X_k$. Define

$$\tau_n = n^{-1} \cdot \min\left\{1 \leq k \leq n : S_k = \max_{j \leq n} S_j\right\}.$$

Prove that τ_n converges to the arcsine law.
What happens if we change the min to max in the definition of τ_n?

5. Let $f : \mathbb{R}^d \to \mathbb{R}$ be harmonic, that is, twice continuously differentiable and $\Delta f = 0$ (where $\Delta f = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}$), and let $B(t)$ be d-dimensional Brownian motion. Assume also that $\mathbb{E}_x|f(B(t))| < \infty$ for each $x \in \mathbb{R}^d$ and every $t \geq 0$.

   (a) Show that $f(B(t))$ is a martingale.

   (b) Consider $d = 2$ and interpret $\mathbb{R}^2$ as the complex plane. Show that for any $\lambda \in \mathbb{R}$ the process $\{e^{i\lambda B(t)}\}_{t \geq 0}$ is martingale, that is, its real and imaginary parts are martingales.

   (c) In the setting of (b), assume that $B(t)$ starts at i and let T be the first time $B(t)$ visits the real axis. Show that $\mathbb{E}e^{i\lambda B(T)} = e^{-|\lambda|}$ and use Lévy's inversion formula (from Prob to Math) to find $B(T)$'s distribution.