Brownian motion — exercise sheet 4

Due Wednesday, June 30th. Notation:

- $\{B_t : t \in [0,\infty)\}$ is standard Brownian motion started at 0 in $\mathbb{R}$.
- $L(t;x)$ is the local time of Brownian motion at time $t \geq 0$ at $x \in \mathbb{R}$, that is,

$$L(t;x) = L_2 - \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{\{[x-\epsilon, x+\epsilon\}}(B_s)ds\,.$$

  Or alternatively by Tanaka's formula $L(t;x) = |B_t - x| - |x| - \int_0^t \operatorname{sign}(B_s - x)dB_s$ a.s.
- For $a \in \mathbb{R}$ we denote $T_a = \inf\{t \geq 0 : B(t) = a\}$.

1. Show that for any continuous function $f : \mathbb{R} \to \mathbb{R}$ with compact support

$$\int_0^t f(B_s)ds = \int_{\mathbb{R}} f(x)L(t;x)dx\,,$$

   almost surely.
2. Let $a \neq 0$. Show that $L(T_a;0)$ has exponential distribution with mean $2|a|$.
3. Find a continuous function $f : [0,1] \to \mathbb{R}$ such that the measure on $C[0,1]$ induced by the process $\{B_t + f(t)\}_{t\in[0,1]}$ is singular to Wiener measure.
4. Let $T > 0$ be fixed. Find

$$\mathbb{E}\Big[(B_T - T)^2 \exp\Big(\int_0^T e^{-s}dB_s\Big)\Big]\,.$$