Complex analysis 2
Homework #1
Due date: March 25, 2019

$\mathbb{C}$ = complex plane; $\mathbb{D} = \{z : |z| < 1\}$.

1. Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function such that $|f'(0)| = 1$ and $|f'(z)| \leq 2$ for all $z \in \mathbb{D}$. Show that the image $f(\mathbb{D})$ contains a disc of radius $3 - \sqrt{8}$. [Remark: points will be given if you can prove this with a constant less than $3 - \sqrt{8}$]

2. Let $\mathcal{F}$ be the family of holomorphic functions $\mathbb{D} \to \mathbb{C}$ that are one-to-one.
   (a) Show that $\mathcal{F}$ is not normal.
   (b) Let $\mathcal{F}_0$ be all the functions of $\mathcal{F}$ that omit the value 0 (that is, $f \in \mathcal{F}_0$ if and only if $f \in \mathcal{F}$ and $0 \notin f(\mathbb{D})$). Show that $\mathcal{F}_0$ is normal.

3. Let $\mathcal{F}_1$ be the family of holomorphic functions $\mathbb{D} \to \mathbb{C}$ that are one-to-one and $f(0) = 0$ and $f'(0) = 1$. Show that $\mathcal{F}_1$ is normal.

4. Show that there exists $\epsilon_0 > 0$ such that for every holomorphic $f : \mathbb{D} \to \mathbb{C}$ that is one-to-one, $f(0) = 0$ and $f'(0) = 1$ satisfies that the image $f(\mathbb{D})$ contains a ball of radius ϵ_0.

5. Show that there exists $\epsilon_0 > 0$ such that for every holomorphic $f : \mathbb{D} \to \mathbb{C}$ with $f'(0) = 1$ satisfies that the image $f(\mathbb{D})$ contains a ball of radius ϵ_0.