1. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is entire and not a translation (i.e., it is not of the form $f(z) = az + b$ for fixed $a, b \in \mathbb{C}$). Show that $f \circ f$ has a fixed point. I.e., show that there exists $z_0 \in \mathbb{C}$ such that $f(f(z_0)) = z_0$.

2. Let $\Omega = \mathbb{D} \setminus \{0\}$. Show that there does not exist $h : \Omega \to \mathbb{R}$ such that $h$ is harmonic in $\Omega$ and $\lim_{z \to 0} h(z) = 0$ and $\lim_{z \to \partial \mathbb{D}} h(z) = 1$.

3. Let $h : \Omega \to \mathbb{R}$ be a harmonic function. Assume that $|h(z)| \leq M$ for all $z \in \Omega$. Show that $|\nabla u(z)| \leq \frac{2M}{\text{dist}(z, \partial \Omega)}$ where $\text{dist}(z, \partial \Omega)$ is the distance of $z$ from $\partial \Omega$ and $|\nabla u(z)|$ is the norm of the gradient of $u$ at $z$ (that is, $|\nabla u(z)| = \sqrt{u_x(z)^2 + u_y(z)^2}$).

4. Let $h : \partial \mathbb{D} \to \mathbb{R}$ be a continuous function and $u : \mathbb{D} \to \mathbb{R}$ defined by the Poisson formula

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |a|^2}{|e^{it} - a|^2} h(e^{it})dt.$$ 

Show that $\lim_{a \to \xi} u(a) = h(\xi)$ for any $\xi \in \partial \mathbb{D}$.

5. Let $h : \partial \mathbb{D} \to \mathbb{R}$ be a continuous function and let $a \in \mathbb{D}$. For each point $e^{i\theta} \in \partial \mathbb{D}$, let $e^{i\theta^*}$ be the point in $\partial \mathbb{D}$ such that $a, e^{i\theta}, e^{i\theta^*}$ lie on a straight line. Show that the harmonic extension $u : \mathbb{D} \to \mathbb{R}$ of $h$ is given by

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\theta^*})d\theta.$$ 

6. Suppose $u : \mathbb{D} \to \mathbb{R}$ is harmonic and $u(z) \geq 0$ for all $z \in \mathbb{D}$. Show that for any $z \in \mathbb{D}$

$$\frac{1 - |z|}{1 + |z|} u(0) \leq u(z) \leq \frac{1 + |z|}{1 - |z|} u(0).$$

7. Suppose that $u : \mathbb{D} \setminus \{0\} \to \mathbb{R}$ is harmonic and that $\lim_{z \to 0} u(z)$ is either $+\infty$ or $-\infty$. Show that $u$ can be written as

$$u(z) = \beta \ln |z| + u_1(z),$$

where $\beta \neq 0$ is real, and $u_1 : \mathbb{D} \to \mathbb{R}$ is harmonic in $\mathbb{D}$. [Hint: show that the residue of $u_x - iu_y$ is real]