Complex analysis 2
Homework #3
Due date: Thursday in class, May 2nd, 2019

$\mathbb{Z}$ = integers; $\mathbb{R}$ = real numbers; $\mathbb{C}$ = complex plane;
$\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ extended complex plane

1. Assume that Ω and Ω_1 are domains in $\mathbb{C}$. Show that if $f\Omega \to \Omega_1$ is analytic and if v is subharmonic and continuous on Ω_1, then $v \circ f$ is subharmonic on Ω. [Note that subharmonic functions are not necessarily differentiable]
2. Let $w \in \mathbb{C}$ be non-zero, and let $\mathbb{Z}w$ be the integer multiples of w. We set an equivalence relation on $\mathbb{C}$ by putting $z_1 \sim z_2$ when $z_1 - z_2 \in \mathbb{Z}w$. Let R be the set of congruence classes of $\sim$. Show that R is a Riemann surface that is conformally equivalent to the punctured plane $\mathbb{C} \setminus \{0\}$.
3. Show that if $f : R \to S$ is a nonconstant analytic map between two Riemann surfaces, then f is open, that is, for any open set $O \subset R$ the set $f(O)$ is open in S.
4. Let $f : R \to \mathbb{C}$ be holomorphic and non-constant on a Riemann surface R. Show that $|f|$ has no local maximum and no positive local minimum on R.
5. Suppose that $f : R \to S$ is a nonconstant analytic map between two Riemann surfaces.
   (a) Show that if R is compact, then $f(R) = S$.
   (b) Use (a) to prove the fundamental theorem of algebra.
6. Let X be compact Riemann surface and let $Y = X \setminus F$ for some finite set F. Show that any non-constant holomorphic map $f : Y \to \mathbb{C}$ has dense image.
7. Show that $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is holomorphic if and only if it is a rational function.
8. Let M, N be two Riemann surfaces and $f : M \to N$ non-constant holomorphic. Show that for any $x \in M$ there exists coordinate maps (U, z) in M and (V, w) in N such that $x \in U$ and $f(x) \in V$ and $F = w \circ f \circ z^{-1} : z(U) \to w(V)$ equals $F(z) = z^k$ for some $k \in \mathbb{N}$.