Complex analysis 2
Homework #4
Due date: Thursday in class, June 6th, 2019

$\mathbb{Z}$ = integers; $\mathbb{R}$ = real numbers; $\mathbb{C}$ = complex plane;
$\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ extended complex plane

1. Let W be a Riemann surface and $b \in W$. Denote by $[\gamma]$ the equivalence class under homotopy of a curve $\gamma : [0,1] \to W$. Let W^* denote the collection of equivalence classes

$$W^* = \{[\gamma] : \gamma \text{ is a curve in } W \text{ with } \gamma(0) = b\}.$$

   If $c \in W$ let B be a small parametric disk in W centered at c and let γ be a curve from b to c. For any point $d \in B$, let σ_d be a curve in B from c to d. Let

$$B^* = \{[\gamma\sigma_d] : d \in B\},$$

   where $\gamma\sigma_d$ is the concatenation of γ followed by σ_d. Denote by $\pi : W^* \to W$ the map $\pi([\gamma]) = \gamma(1)$ so that π is a one-to-one map of B^* onto B. The sets B^*, for every disk B and any γ from b to c, are declared to be open sets and form a basis for the topology of W^*.

   Show that W^* is a simply connected Riemann surface that covers W. [W^* is called the *universal cover of* W]

2. Without using the uniformization theorem, show that if S is a simply connected Riemann surface that covers W, then S is conformally equivalent to W^*.

3. Show that if Green's function exist for a Riemann surface W_1, and $W_2 \subset W_1$ is a subsurface of W_1, then Green's function exist for W_2 and $g_{W_2} \leq g_{W_1}$. Give a natural definition for the word "subsurface" so that this statement holds.

4. Let W be a Riemann surface without a Green's function. Show any any positive harmonic function on W is constant.

5. Prove that Green's function with pole at p_0 exist on a Riemann surface W if and only if there exists a non-constant positive harmonic function on $W \setminus \{p_0\}$.