Probability in 2D problem set 2
Due date: December 26, 2016

1. Let G be a transitive infinite connected graph and let ρ be an arbitrary vertex of G. For an integer $r > 0$ denote by $\partial B_\rho(r)$ the set of vertices which have graph distance r to ρ. Consider bond percolation on G with $p_c = p_c(G)$. Show that
$$\mathbb{E}\Big|\{v \in \partial B_\rho(r) : \rho \longleftrightarrow v\}\Big| \geq 1\,.$$

2. Consider bond percolation in the box $\{-n, \ldots, n\}^3 \subset \mathbb{Z}^3$ with $p = p_c(\mathbb{Z}^3)$. Show that there exists constants c_1, c_2 such that for all n
$$\mathbf{P}\big((0,0,0) \longleftrightarrow (n,n,n)\big) \geq c_1 e^{-c_2 \log^2(n)}\,.$$

3. Consider bond percolation in the slab $\{1, \ldots, n\} \times \{1, \ldots, n\} \times \{1, \ldots, n/3\} \subset \mathbb{Z}^3$ with $p = p_c(\mathbb{Z}^3)$. Denote by p_n the probability that there is an open path crossing the slab in the third coordinate. Show that there exists a constant $c > 0$ such that $p_n \geq c$ for all n.

4. Let $\mathrm{Maj}_1 : \{0,1\}^3 \to \{0,1\}$ be the majority boolean function on 3 bits. Recursively define for $n \geq 2$ the function $\mathrm{Maj}_n : \{0,1\}^{3^n} \to \{0,1\}$ by setting
$$\begin{aligned} y_1 &= (x_1, \ldots, x_{3^{n-1}}) \\ y_2 &= (x_{3^{n-1}+1}, \ldots, x_{2\cdot 3^{n-1}}) \\ y_3 &= (x_{2\cdot 3^{n-1}+1}, \ldots, x_{3^n}) \end{aligned}$$
and putting
$$\mathrm{Maj}_n(x_1, \ldots, x_{3^n}) = \mathrm{Maj}_1(\mathrm{Maj}_{n-1}(y_1), \mathrm{Maj}_{n-1}(y_2), \mathrm{Maj}_{n-1}(y_3))\,.$$
Is Maj_n asymptotically noise-sensitive? Prove your claim.

5. A boolean function $f : \{0,1\}^n \to \{0,1\}$ is called *transitive* if for any $i, j \in [n]$ there exists a permutation $\sigma \in S_n$ of $[n]$ such that $\sigma(i) = j$ and such that for any $x \in \{0,1\}^n$ we have that $f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Show that there exists some constant $c > 0$ (independent of n) such that any randomized adaptive algorithm A determining a balanced (i.e., $\mathbb{E}f = 1/2$) transitive boolean function f has revealment $\delta_A \geq c n^{-1/2}$.