Random walks on random fractals — list of notation and theorems for week 2

$\{x \leftrightarrow y\}$ – x is connected to y by an open path.
$\{x \overset{r}{\leftrightarrow} y\}$ – x is connected to y by an open path of length at most r.
$\{x \overset{=r}{\leftrightarrow} y\}$ – x is connected to y by an open path, the length of the shortest open path equals r.
$\{x \overset{r}{\leftrightarrow} y\} \circ \{w \overset{\ell}{\leftrightarrow} z\}$– there exists two **edge disjoint** open paths of length at most r and ℓ connecting x to y and w to z, respectively.

$$\mathcal{C}(0) := \{z : 0 \leftrightarrow z\} \qquad B(0,r) := \{z : 0 \overset{r}{\leftrightarrow} z\} \qquad \partial B(0,r) := \{z : 0 \overset{=r}{\leftrightarrow} z\}\,.$$

Theorem 0 (BK-inequality) *(v.d. Berg, Kesten '85) On any graph G and any $p \in [0,1]$ we have*

$$\mathbf{P}_p\big(\{x \overset{r}{\leftrightarrow} y\} \circ \{w \overset{\ell}{\leftrightarrow} z\}\big) \le \mathbf{P}_p(x \overset{r}{\leftrightarrow} y)\mathbf{P}_p(w \overset{\ell}{\leftrightarrow} z)\,.$$

Theorem A (Triangle condition) *(Hara, Slade '90) For $\mathbb{Z}^d$ with $d \ge 19$ we have*

$$\sum_{x,y\in\mathbb{Z}^d} \mathbf{P}_{p_c}(0 \leftrightarrow x)\mathbf{P}_{p_c}(x \leftrightarrow y)\mathbf{P}_{p_c}(y \leftrightarrow 0) < \infty\,.$$

Note: we will use the "open" triangle condition: there exists $K \ge 1$ such that if u, v are vertices such that $|u-v| \ge K$, then

$$\sum_{x,y\in\mathbb{Z}^d} \mathbf{P}_{p_c}(u \leftrightarrow x)\mathbf{P}_{p_c}(x \leftrightarrow y)\mathbf{P}_{p_c}(y \leftrightarrow v) \le \frac{1}{10}\,.$$

Theorem B *(Aizenman, Barsky '91) For $\mathbb{Z}^d$ with $d \ge 19$ we have*

$$\mathbf{P}_{p_c}(|\mathcal{C}(0)| \ge n) \approx \frac{1}{\sqrt{n}}\,.$$

Theorem C *(Hara, v.d. Hofstad, Slade '03 & Hara '08) For $\mathbb{Z}^d$ with $d \ge 19$ we have*

$$\mathbf{P}_{p_c}(x \leftrightarrow y) \approx |x-y|^{2-d}\,.$$

Theorem D *(v.d. Hofstad, Járai '04) For $\mathbb{Z}^d$ with $d \ge 19$ the limit*

$$\mathbf{P}_{\mathrm{IIC}}(F) := \lim_{|x|\to\infty} \mathbf{P}_{p_c}(F \mid 0 \leftrightarrow x)\,,$$

exists for any cylinder event F.

For a subgraph $G \subset E(\mathbb{Z}^d)$ perform percolation with parameter $p_c(\mathbb{Z}^d)$ and write $\partial B(0,r;G)$ for the set of vertices x such that $0 \overset{=r}{\leftrightarrow} x$. When $G = \mathbb{Z}^d$ we have $\partial B(0,r;G) = \partial B(0,r)$. Lastly, define

$$\Gamma(r) = \sup_{G\subset E(\mathbb{Z}^d)} \mathbf{P}_{p_c(\mathbb{Z}^d)}\big(\partial B(0,r;G) \ne \emptyset\big)\,.$$

Theorem 1 *For $\mathbb{Z}^d$ with $d \ge 19$ there exists a constant $C > 0$ such that*

(a) $\mathbb{E}_{p_c}|B(0,r)| \le Cr$

(b) $\Gamma(r) \le Cr^{-1}$.

Remark. The reason for defining $\Gamma(r)$ is that the event $\partial B(0,r) \ne \emptyset$ is not monotone with respect to adding edges. *Open problem:* Show that the supremum in the definition of $\Gamma(r)$ is attained when $G = \mathbb{Z}^d$.

Theorem 2 *For any $\epsilon > 0$ there exists $C \ge 1$ such that for all $r \ge 1$*

$$\mathbf{P}_{\mathrm{IIC}}\Big(\frac{|B(0,r)|}{r^2} \in [C^{-1}, C] \text{ and } \frac{R_{\mathrm{eff}}(0,\partial B(0,r))}{r} \in [C^{-1}, 1]\Big) \ge 1-\epsilon\,.$$

Theorem 3 *For any $\epsilon > 0$ there exists $C \ge 1$ such that for all $r \ge 1$*

$$\mathbf{P}_{\mathrm{IIC}}\Big(\frac{|\mathbb{E}\tau_r|}{r^3} \in [C^{-1}, C] \text{ and } r^2 p_{r^3}(0,0) \in [C^{-1}, C]\Big) \ge 1-\epsilon\,.$$

Random walks on random fractals — exercises for week 2

1. Show that Theorem C implies the open triangle condition, that is, for any $\epsilon > 0$ there exists $K \geq 1$ such that if $|u - v| \geq K$ then

$$\sum_{x,y \in \mathbb{Z}^d} \mathbf{P}_{p_c}(u \leftrightarrow x)\mathbf{P}_{p_c}(x \leftrightarrow y)\mathbf{P}_{p_c}(y \leftrightarrow v) \leq \epsilon \, .$$

2. Show that Theorem 1 implies that $\mathbf{P}_{p_c}(|\mathcal{C}(0)| \geq n) \leq cn^{-1/2}$ (that is, Theorem 1 implies the upper bound of Theorem B).

3. Assume that there exists an $C > 0$ such that for all $\epsilon > 0$ we have $\mathbb{E}_{p_c - \epsilon}|\mathcal{C}(0)| \leq C\epsilon^{-1}$. Show that this implies part (a) of Theorem 1. (Remark: the assumption is known to hold for $\mathbb{Z}^d$ with $d \geq 19$ so this gives an alternate proof of part (a) of Theorem 1 and is due to Artem Sapozhnikov).

4. Prove the corresponding lower bounds for Theorem 1. That is, that there exists a constant $c > 0$ such that $\mathbb{E}_{p_c}|B(0,r)| \geq cr$ and that $\Gamma(r) \geq cr^{-1}$.

5. Let $S_r = \{x \in \mathbb{Z}^d : |x| \geq r\}$, show that $\mathbf{P}_{p_c}(0 \leftrightarrow S_r) \geq cr^{-2}$ for some constant $c > 0$.

6. Show that there exists some constant $c > 0$ such that for all $A \geq 1$ we have

   (a) $\mathbf{P}_{\mathrm{IIC}}\big(|B(0,r)| \geq Ar^2\big) \leq e^{-cA}$.

   (b) $\mathbf{P}_{\mathrm{IIC}}\big(|B(0,r)| \leq A^{-1}r^2\big) \leq e^{-cA}$.

   [Hint for part (a): bound the moments $\mathbb{E}_{p_c}|B(0,r)|^k$]

7. Show that there exists some constant $c > 0$ such that for all $A \geq 1$ we have

$$\mathbf{P}_{\mathrm{IIC}}\big(R_{\mathrm{eff}}(0, \partial B(0,r)) \leq A^{-1}r\big) \leq e^{-cA^c} \, .$$

8. Conclude from the last two problems that $\mathbf{P}_{\mathrm{IIC}}$-almost-surely $d_f = 2$ and $d_s = 4/3$, where

$$d_f = \lim_{r \to \infty} \frac{\log |B(0,r)|}{\log r} \qquad d_s = -2 \lim_{t \to \infty} \frac{\log p^t(0,0)}{\log t} \, .$$

9. For a vertex x in the IIC write $d_{\mathrm{IIC}}(0,x)$ for the graph distance between 0 and x in the IIC. Let $\{X_n\}$ be the simple random walk on the IIC in $\mathbb{Z}^d$ with $d \geq 19$. Show that $\mathbf{P}_{\mathrm{IIC}}$-almost-surely

$$\lim_{n \to \infty} \frac{\log d(0, X_n)}{\log n} = 1/3 \, , \qquad \{X_n\}\text{-almost-surely} \, .$$