ON THE NUMBER OF SOLUTIONS TO THE LINEAR COMPLEMENTARITY PROBLEM

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Given an $n \times n$ matrix A and an n-dimensional vector q let $N(A, q)$ be the cardinality of the set of solutions to the linear complementarity problem defined by A and q. It is shown that if A is nondegenerate then $N(A, q) + N(A, -q) \le 2^n$, which in turn implies $N(A, q) \le 2^n - 1$ if A is also a Q-matrix.

It is then demonstrated that $\min_{q \ne 0} N(A, q) \le 2^{n-1} - 1$, which concludes that the complementary cones cannot span $\mathbf{R}^n$ more than $2^{n-1} - 1$ times around. For any n, an example of an $n \times n$ nondegenerate Q-matrix spanning all $\mathbf{R}^n$, but a subset of empty interior, $2^{[n/3]}$ times around is given.

Given a square real matrix A of order n and an n-dimensional vector q, the Linear Complementarity Problem, (LCP), denoted by (q, A), is to find n-dimensional vectors x and y satisfying

$$y = Ax + q, \quad x \ge 0, y \ge 0; \quad x'y = 0. \tag{1}$$

(See [3,5,6,7] for the bibliography and history of the LCP.)

Letting $N(A, q)$ be the cardinality of the set of solutions to (q, A) we study the quantities $N^*(A) = \max_{q \in \mathbf{R}^n} N(A, q)$ and $N_*(A) = \min_{0 \ne q \in \mathbf{R}^n} N(A, q)$ and establish bounds on both. The latter quantity has the following geometrical interpretation. If we consider the 2^n complementary cones generated by the columns of $[I, -A]$, (see [8]), then $N_*(A)$ is the number of times $\mathbf{R}^n$ is spanned around by these cones.

This study shows that this (spanning) number is bounded above by $2^{n-1} - 1$ if the matrix A is nondegenerate, i.e. all its principal minors are nonzero. For any n, an example of an $n \times n$ nondegenerate matrix

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spanning all $\mathbf{R}^n$, but a subset of empty interior, $2^{[n/3]}$ times around is given.

It is also demonstrated that the nondegeneracy of A implies $N(A, q) + N(A, -q) \leq 2^n$ for all q. Thus, if A is a Q-matrix (i.e. $N(A, q) \geq 1$ for all q) then $N^*(A) \leq 2^n - 1$.

This work was motivated by a conjecture due to Ingleton [4] and Censor [1]. Studying nondegenerate matrices Ingleton showed that if for some q^* (q^*, A) has a unique solution (x^*, y^*) and $x^* + y^*$ is strictly positive, then A is a Q-matrix. (Extensions of this result appear in [5,8,9,10]). Ingleton conjectured that under the above conditions the maximal number of solutions to (q, A) is $2^n - 1$, provided that $N(A, \bar{q}) > 1$ for some $\bar{q}$.

The conjecture was studied and partially resolved by Censor, [1], who also raised the following general question: Given an $n \times n$ nondegenerate Q-matrix A and supposing that $N(A, \bar{q}) > 1$ for some $\bar{q}$, is the maximal number of solutions to (1) $2^n - 1$ for all q in $\mathbf{R}^n$?

The following result demonstrates that $2^n - 1$ is indeed a valid bound. (We later show that the bound is not necessarily attained.)

Lemma 1. *Let A be an $n \times n$ matrix such that $N(A, q) \geq 1$ for all $q \leq 0$. Then for any q, $N(A, q) < \infty$ implies $N(A, q) \leq 2^n - 1$.*

Proof. For any $I \subseteq \{1, 2, ..., n\}$ consider the (complementary) cone generated by the n columns $-A_{\cdot i}$, $i \in I$ and e_i, $i \in \bar{I}$, where $A_{\cdot i}$ is the ith column of A and e_i is the ith unit vector. Note that the correspondence between subsets of $\{1, 2, ..., n\}$ and the complementary cones is not necessarily one to one.

It is obvious that if there are two different solutions to (2)

$$y = Ax + q, \quad x \geq 0, \quad y \geq 0, \quad x'y = 0,$$

$$x_i > 0 \; i \in I \quad \text{and} \quad x_i = 0 \quad i \notin I, \tag{2}$$

for some $I \subseteq \{1, ..., n\}$ then $N(A, q)$ is not finite. (Consider convex combinations of these two solutions.)

Hence, if $N(A, q)$ is finite each set $I \subseteq \{1, 2, ..., n\}$ contributes at most one solution to (q, A) and $N(A, q) \leq 2^n$. If $q \not> 0$, then the cone defined by $I = \emptyset$ does not contribute a solution and $N(A, q) \leq 2^n - 1$. Note that $N(A, 0)$ is finite only if $N(A, 0) = 1$. Consider now a nonzero, nonnegative q. By the lemma's assumption $(-q, A)$ has a solution.

Defining a cone to be nondegenerate if it has a nonempty interior, we observe that the union of all the degenerate complementary cones and the (proper) faces of the nondegenerate cones is closed and no where dense in $\mathbf{R}^n$. Thus there exists a sequence $\{q^k\}$, converging to $-q$ where $\forall k$ $q^k \leqslant 0$ and q^k belongs to a nondegenerate cone. Since there is a finite number of complementary cones we can assume without loss of generality (choose a subsequence if necessary) that $\forall k$ q^k belongs to the same nondegenerate cone. Hence $-q$ is in that cone. It is then clear that q is not contained in this nondegenerate cone, since otherwise we would have that the n generators are linearly dependent -- a contradiction to the nondegeneracy assumption. Hence $N(A, q) \leqslant 2^n - 1$.

From Lemma 1, $N^*(A) \leqslant 2^n - 1$ for any nondegenerate Q-matrix A. Censor, [1], provided an example of an $n \times n$ nondegenerate Q-matrix, (which we denote by A_n), such that $N^*(A_n) = 2^n - 1$. Murty, [8], illustrated that the matrix

$$M = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \tag{3}$$

is a nondegenerate Q-matrix and $N^*(M) = 4$. Combining the above two illustrations and using the direct sum operation we show that for each n $2^n - 1$ is not always a sharp bound.

Recalling that the direct sum of the square matrices A and B, $A \oplus B$, is given by

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

we notice that

$$N(A \oplus B, q) = N(A, q^1) N(B, q^2), \tag{4}$$

where (q^1, q^2) is the corresponding partition of q.

Lemma 2. *Given nonnegative integers n, r, t, such that $3t + r \leqslant n$ there exists an $n \times n$ nondegenerate Q-matrix A such that $N^*(A) = 4^t \max\{1, 2^r - 1\}$.*

Proof. Define

$$A = \underbrace{M \oplus \ldots \oplus M}_{t} \oplus A_r \oplus I_{n-3t-r},$$

where M is given by (3), A_r is the above mentioned matrix due to Censor [1] and I_{n-3t-r} is the identity of order $n - 3t - r$.

Having established an upper bound on $N(A, q)$, we next turn to the lower bound question and prove that if A is an $n \times n$ nondegenerate Q-matrix then $\min_{q \neq 0} N(A, q) \leqslant 2^{n-1} - 1$, provided $n \geqslant 2$.

Lemma 3. *Let A be an $n \times n$ matrix and q in R^n. If $N(A, q) + N(A, -q) < \infty$, then $N(A, q) + N(A, -q) \leqslant 2^n$.*

Proof. If $q = 0$, then $N(A, 0) < \infty$ implies $N(A, 0) = 1$. Thus consider $q \neq 0$. Let $I \subseteq \{1, 2, ..., n\}$, then we show that the complementary cone corresponding to I does not contain both q and $-q$. This will prove the lemma. Observe first that if the cone corresponding to I is nondegenerate, i.e. has an interior, then $q \neq 0$ implies that the cone cannot contain both q and $-q$. Suppose that there exists a cone C containing q and $-q$. Let B be an $n \times n$ matrix corresponding to the n generators of C, then $Bx^1 = q$ and $Bx^2 = -q$ for some $x^1 \geqslant 0$ and $x^2 \geqslant 0$. Hence $\forall t \geqslant 0$, $B[(1 + t)x^1 + tx^2] = q$ and there exists a (complementary) ray of solutions – a contradiction to $N(A, q) < \infty$.

Note that Lemma 3, which has been proved independently of Lemma 1, also implies that $N(A, q) \leqslant 2^n - 1$ for any nondegenerate $n \times n$ Q-matrix. Another consequence is that $\min_{q \neq 0} N(A, q) \leqslant 2^{n-1}$.

We now refine this bound to $2^{n-1} - 1$, applying a result due to Murty [8].

Theorem 4. *Let A be an $n \times n$ nondegenerate matrix. Then $\min_{q \neq 0} N(A, q) \leqslant 2^{n-1} - 1$ when $n \geqslant 2$.*

Proof. From Lemma 3 it follows that $\min_{q \neq 0} N(A, q) \leqslant 2^{n-1}$. Suppose that for all $q \neq 0$ $N(A, q) = 2^{n-1}$. We then apply Theorem (7.2) of [8] to have $N(A, q) = 1$ for all q. Hence the proof is complete.

Theorem 4 concludes that the complementary cones cannot span the space $\mathbf{R}^n$ more than $2^{n-1} - 1$ times around. This bound is trivially attained for 2×2 Q-matrices.

Finally we show that for any $n \geqslant 3$ there exists an $n \times n$ nondegenerate Q-matrix A such that the complementary cones defined by the columns of $[I, -A]$ span $\mathbf{R}^n - R(A)$ $2^{[n/3]}$ times around, where $R(A)$ is a subset of $\mathbf{R}^n$ with empty interior.

To define $R(A)$ precisely, we recall that q in $\mathbf{R}^n$ is said to be nondegenerate with respect to the $n \times n$ matrix A if for any (x, y) solving (q, A), $x + y$ is a strictly positive vector (i.e. q is neither contained in a complementary cone with no interior nor in a proper face of a nondegenerate cone). q is degenerate if it is not nondegenerate. $R(A)$ is defined as the set of vectors in $\mathbf{R}^n$ that are degenerate with respect to A.

We need the following lemma for our discussion.

Lemma 5. *Let A be an $n \times n$ nondegenerate matrix and consider q in $\mathbf{R}^n$ which is not contained in any face of dimension less than $n - 1$ of a complementary cone defined by the columns of $[I, -A]$. Then, $\forall \epsilon > 0$ there exist nondegenerate vectors P^1 and P^2 such that*

$$\| P^1 - q \| < \epsilon, \qquad \| P^2 - q \| < \epsilon,$$

$$N(A, q) = \tfrac{1}{2} [N(A, P^1) + N(A, P^2)] \,.$$

Proof. If q itself is nondegenerate choose $P^1 = P^2 = q$.

Suppose that q is contained in the intersection of k different $(n - 1)$-dimensional faces $C_1, ..., C_k$ of complementary cones but not in any face of a lower dimension. Choose $\epsilon > 0$ such that $S(q, \epsilon) = \{x \mid \| x - q \| \leq \epsilon\}$ is contained in any complementary cone having q in its interior, and such that each complementary cone having one of $C_1, ..., C_k$ as its $(n - 1)$-dimensional face contains the corresponding "half" ball chopped off by that face. ϵ is well defined since each complementary cone is nondegenerate and q is not in any face of dimension less than $n - 1$.

Let $u \in \mathbf{R}^n$, $\| u \| < \epsilon$, such that $q + u$ and $q - u$ are nondegenerate vectors. We then have that $P^1 = q + u$ and $P^2 = q - u$ are on opposite sides of each C_i, $i = 1, ..., k$. Also, P^1 and P^2 are contained in any complementary cone having q in its interior.

Consider the ith face C_i, and let t_i be the number of different complementary cones having C_i as their face. Using the assumption that q is not on a face of dimension less than $n - 1$ we may suppose, without loss of generality, that $\bar{t}_i$ cones, $0 \leq \bar{t}_i \leq t_i$ contain P^1 and $t_i - \bar{t}_i$ contain P^2.

Let $\bar{T}_i(T_i - \bar{T}_i)$ be the cardinality of the set of solutions to (P^1, A) $((P^2, A))$ corresponding to the above $\bar{t}_i(t_i - \bar{t}_i)$ cones. (Note that we introduce T_i and $\bar{T}_i$ since the correspondence of subsets of $\{1, 2, ..., n\}$ to

the different complementary cones is not injective.) Note that T_i is even by the nondegeneracy assumption. Thus the t_i cones contribute $T_i/2$ solutions to (q, A), $\bar{T}_i$ solutions to (P^1, A) and $T_i - \bar{T}_i$ solutions to (P^2, A). Noting that P^1 and P^2 are on opposite sides of each C_i, $i = 1, ..., k$ and denoting by r the number of nondegenerate solutions to (q, A) we have

$$N(A, P^1) = r + \sum_{i=1}^{k} \bar{T}_i, \qquad N(A, P^2) = r + \sum_{i=1}^{k} (T_i - \bar{T}_i),$$

$$N(A, q) = r + \sum_{i=1}^{k} \frac{T_i}{2}.$$

Hence $N(A, q) = \frac{1}{2}[N(A, P^1) + N(A, P^2)]$ and the proof is complete.

Dealing with the 3×3 matrix M defined in (3), Murty [8] claims that $N(M, q)$ is positive and even for all nondegenerate q. Lemma 5 and a simple inspection of the six generators of $[I, -M]$ show that $N(M, q) \geq 2$ for all $q \neq 0$. Clearly $N(M, 0) = 1$.

Let M be given by (3) and define

$$M^{(j)} = \underbrace{M \oplus ... \oplus M}_{j}.$$

If q is in $\mathbf{R}^{3j}$, then we have $q = (q^1, q^2, ..., q^j)$, where q^t, $1 \leq t \leq j$, is in $\mathbf{R}^3$. We then observe that for $q \in \mathbf{R}^{3j}$ such that $q^t \neq 0$, $t = 1, ..., j$ $N(M^{(j)}, q) \geq 2^j$, and that $\{q \mid q^t = 0$ for some $1 \leq t \leq j\}$ is (strictly) contained in $R(M^{(j)})$, the set of degenerate vectors corresponding to $M^{(j)}$. Thus the following theorem is implied.

Theorem 6. *For any $n \geq 3$ there exists an $n \times n$ nondegenerate Q-matrix A such that $N(A, q) \geq 2^{\lfloor n/3 \rfloor}$ for any q in $\mathbf{R}^n - B$, where B is a subset of $R(A)$, the set of degenerate vectors corresponding to A.*

Finally, while noting that for $j \geq 2$, $N(M^{(j)}, e_1) = 2$, where $e_1 = (1, 0, ..., 0)'$ we conclude that Lemma 5 does not hold for all $q \neq 0$. In fact, the above illustrates that if q is contained in a face of dimension less than $n - 1$, where n is the order of the matrix A, the existence of nondegenerate vectors P^1 and P^2 such that $N(A, P^1) \leq N(A, q) \leq N(A, P^2)$ is not guaranteed.

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