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ON THE CORE OF A TRAVELING SALESMAN COST ALLOCATION GAME

Arie TAMIR

New York University, Washington Square, New York, NY 10003 and Tel Aviv University, Ramat-Aviv, Tel-Aviv, Israel

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Let $G=(V, E)$ be a connected undirected graph with positive edge lengths. Let $V=\{0\} \cup N$, where $N=\{1,\dots,n\}$. Each node in N is identified as a customer, and 0 is the home location of a traveling salesman or repairman who serves the customers in N. Each subset of customers S can hire the repairman to serve its members only. In that case the cost incurred by S, $c(S)$, is the minimum length of a tour traversed by the repairman who starts at node 0, visits each node in S at least once and returns to 0. We consider the core of the cooperative cost allocation game $(N; c)$ defined by the cost function $c(S)$, $S \subseteq N$. We show that the core can be empty even if G is series parallel by presenting the unique minimal counter example for such graphs. We then use a recent result of Fonlupt and Naddef, and prove that the core is nonempty for a class of graphs that properly contains the subclass of cycle trees, i.e. graphs which have no edge included in more than one simple cycle.

graph theory * traveling salesman * cost allocation problem

This paper is motivated by the following problem. A repairman is hired by several customers to visit and serve them. He starts from his home city, visits each customer and returns home. The total cost of his trip must be paid by the customers. The problem is to find a fair or a stable allocation of the total cost among the customers.

The model is formulated as a cooperative game and its core is discussed.

Let $G=(V, E)$ be a finite, loopless, connected undirected graph with node set V and edge set E. Let $N=\{1,\dots,n\}$ and suppose that $V=\{0\} \cup N$. Each edge e in E is associated with a nonnegative length d_e. Each node i in N is identified as a customer, and 0 is the home location of a traveling salesman or repairman who serves the customers in N. Each subset of customers $S \subseteq N$ may form a coalition, and hire the repairman to serve its members only. In that case the cost incurred by the coalition S, $c(S)$, is the minimum length of a tour traversed by the repairman who starts at node 0, visits each node in S at least once and returns to 0. A tour of $\{0\} \cup S$ is a multisubset of E, (an edge may appear more than once), which induces a connected subgraph of G, meets each node in $\{0\} \cup S$ at least once, and can be partitioned into edge disjoint cycles in G. The length of a tour is the sum of the lengths of the edges in the multisubset. (From the Eulerian property no edge will appear more than twice in a minimum tour.)

It is obvious that the cost function $c(S)$, defined on the power set of N is subadditive, i.e. $c(S_1)+c(S_2) \geq c(S_1 \cup S_2)$ for every pair of subsets S_1, S_2 in N. Thus, there is an incentive for the customers to unite, form a grand coalition and hire the repairman to visit all of them in a single tour. A critical and natural question is whether there exists a 'stable' allocation of the total cost, $c(N)$, among the customers that gives no coalition $S \subseteq N$ the incentive to split off and act on its own. Formally, we refer to the core of the cooperative game $(N; c)$ defined by the cost function $c(S)$, $S \subseteq N$.

A vector $x=(x_1,\dots,x_n)$ is a core allocation of the game $(N; c)$ if

$$\sum_{i \in S} x_i \leq c(S) \quad \text{for all } S \subseteq N$$

and

$$\sum_{i \in N} x_i = c(N). \tag{1}$$

The core is the set of all core allocations. The existence of a core allocation for the above game is stated as an important open problem in [3,10]. (Note that in [3,10] the above game is equivalently defined on the complete graph $\bar{G}=(V, \bar{E})$, obtained from G by connecting each pair of nodes in V by an edge, and letting the length of that edge be the shortest distance between its respective end nodes in G.) In this work we will present several examples where the core can be empty including one with $n=6$ which is minimal in a certain respect.

If the graph G is a tree the above traveling salesman cost allocation game coincides with the minimum cost spanning tree game, which is known to possess a core allocation, see [6,7,8]. In fact for this case even the nucleolus of the core can be computed in polynomial time, [9]. We extend this existential result on tree graphs to a wider class of graphs by using the recent result in [4] which characterizes the integer polyhedron of the traveling salesman problem defined on graphs in this class.

We start by discussing games which have no core allocation. Let $G=(V, E)$ be such that all its edges are of unit length. Suppose that G has a hamiltonian cycle, i.e. a simple cycle which meets each node in V. Then, it is easy to verify that the vector which equally allocates the total cost, $c(N)=(n+1)$, among the n customers is a core allocation. Thus, suppose that G is hypohamiltonian, i.e. G has no hamiltonian cycle but each subgraph of G obtained by deleting any node of V has a hamiltonian cycle. We claim that the core of G is empty.

First we note that for each j in N, $c(N-\{j\})=n$. Also $c(N)=(n+2)$. If $x=(x_1,\ldots,x_n)$ were a core allocation we would have

$$\sum_{\substack{i=1\\ i\neq j}}^{n} x_i \leqslant n \quad \text{for all } j \text{ in } N. \tag{2}$$

Summing (2) over all $j\in N$, and equating

$$\sum_{i=1}^{n} x_i \text{ to } c(N)$$

yields the following contradiction if $n\geqslant 3$:

$$n^2 \geqslant (n-1)\sum_{i=1}^{n} x_i = (n-1)c(N)$$
$$= (n-1)(n+2).$$

It is known, [1], that a smallest hypohamiltonian graph has 10 nodes and is isomorphic to the Petersen graph (see Figure 1). The minimality of the Petersen graph as a hypohamiltonian graph might hint that this is a minimal example for an empty core among all graphs with unit edge lengths. This is not the case. Consider the graph in Figure 2. Suppose that the core of the traveling salesman game of this example was not empty. Then due to the symmetry of the model there would be a core allocation $x=(x_1, x_2, x_3, x_4, x_5, x_6)$ with $x_1=x_5=x_6$ and $x_2=x_3=x_4$. It is easily shown that $c(N)=8$ and $c(\{1, 2, 4, 5\})=5$. Hence, x should satisfy

$$3x_1+3x_2=8,$$
$$2x_1+2x_2\leqslant 5.$$

The following contradiction is derived:

$$8=3(x_1+x_2)\leqslant \tfrac{3}{2}(2x_1+2x_2)\leqslant \tfrac{15}{2}.$$

It will later follow that the example given in Figure 2 is minimal among all graphs with unit edge lengths.

It has already been mentioned above that if the graph G is a tree the core of the game is non-empty. Is there a larger class of graphs that share this property? An almost standard extension of tree graphs, often used in combinatorial optimiza-

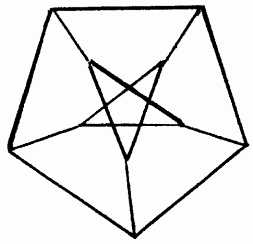

Fig. 1. Petersen graph.

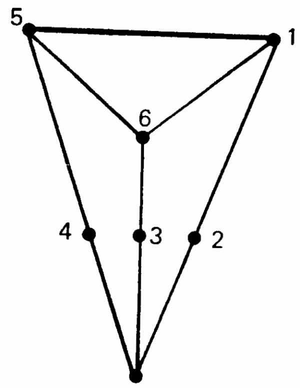

Fig. 2.

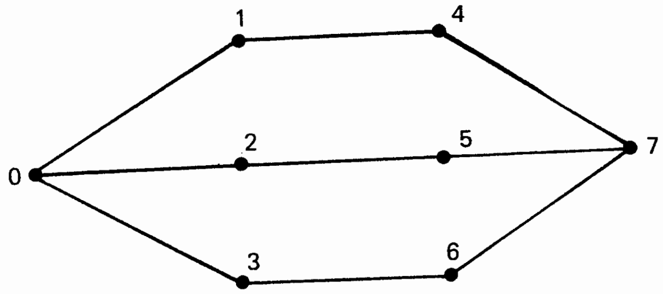

Fig. 3.

tion, is the class of series parallel graphs. This is the class of graphs that do not contain K_4, the complete graph on 4 nodes, as a minor, [2]. Recall that H is a minor of a graph G if H can be obtained from G by a sequence of node deletions, edge deletions and edge contractions.

We will next show a series parallel graph with 8 nodes whose cost allocation game has an empty core. (Note that the graph in Figure 2 which has 7 nodes is not series parallel.) Consider the graph in Figure 3. Using the symmetry of the graph, if the core of the traveling salesman game were not empty there would be a core allocation $x=(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ with $x_1=x_2=x_3$ and $x_4=x_5=x_6$. We easily verify that $c(N)=10$ and $c(\{1, 2, 4, 5, 7\})=6$. Hence, x should satisfy

$$3(x_1+x_4)+x_7=10,$$

$$2(x_1+x_4)+x_7 \leqslant 6.$$

The subadditivity of the cost function implies that $x \geqslant 0$. ($x_i \geqslant c(N)-c(N-\{i\}) \geqslant 0$ for each $i \in N$.)

Therefore we obtain the following contradiction:

$$\begin{aligned} 6 &\geqslant 2(x_1+x_4)+x_7 \geqslant 2(x_1+x_4) \\ &= 2\left[(3(x_1+x_4)+x_7)-(2(x_1+x_4)+x_7)\right] \\ &\geqslant 2[10-6]=8. \end{aligned}$$

Having demonstrated several counter examples we finally extend the existential result on tree graphs to a wider class of graphs by using the result in [4]. (The edges are not restricted anymore to have a unit length.)

To facilitate the discussion we introduce the following notation. Consider a subset of nodes $T \subseteq N$. $\delta(T)$ will denote the set of edges in E with one endpoint in T and the second in $\{0\} \cup (N-T)$.

Consider a coalition of customers $S \subseteq N$. Then $c(S)$ is given by

$$c(S) = \text{Minimum} \sum_{e \in E} d_e x_e$$

subject to

$$\sum_{e \in \delta(\{j\})} x_e \quad \text{is even for all } j \in N, \tag{3}$$

$$\sum_{e \in \delta(T)} x_e \geqslant 2 \quad \text{for all } T \subseteq N, \text{ such that } S \cap T \neq \emptyset, \tag{4}$$

$$x_e \quad \text{is nonnegative and integer for all } e \in E. \tag{5}$$

Consider now a modified game $(N; \bar{c})$ with the following cost function $\bar{c}(S)$, $S \subseteq N$:

$$\bar{c}(S) = \text{Minimum} \sum_{e \in E} d_e x_e$$

subject to

$$\sum_{e \in \delta(T)} x_e \geqslant 2 \quad \text{for all } T \subseteq N \text{ such that } S \cap T \neq \emptyset,$$

$$x_e \geqslant 0 \quad \text{for all } e \in E.$$

It is obvious that $\bar{c}(S) \leqslant c(S)$ for all $S \subseteq N$. Furthermore, it is shown in [5,11] that the modified game $(N; \bar{c})$ can be viewed as a network design game and therefore it has a nonempty core. If $G=(V, E)$ is a graph such that $\bar{c}(N)=c(N)$ then it follows that the nonempty core of the game $(N; \bar{c})$ is included in the core of the game $(N; c)$. The recent paper of Fonlupt and Naddef [4], provides structural sufficient conditions on a graph G which ensure the equality $\bar{c}(N)=c(N)$. These conditions require that G contains no minor which is isomorphic to one of three graphs: the two given in Figures 2 and 3 and the graph of Figure 4. (For example, the conditions are satisfied if G is a cycle tree, i.e. no edge is contained in more than one simple cycle.) If G satisfies the above conditions we can use the polynomial formulation of the network design game in [11] and compute a core allocation by solving a linear program of polynomial (in n) dimensions. (For a cycle tree a core allocation is easy to compute in linear time

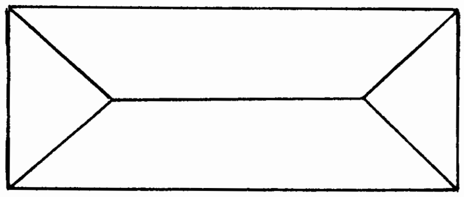

Fig. 4.

by decomposing the game. For each biconnected component $(BC)_j$, i.e. a simple cycle or a bridge edge, consider the game restricted to the node set of $(BC)_j$ and let v_j, the closest node to node 0 in $(BC)_j$, be the home location for this subgame. There is a core point of this subgame. There is a core point of this subgame where at most two nodes in $(BC)_j$ are allocated positive costs.)

An immediate corollary of the above is that if G has at most 5 nodes, $(n \leqslant 4)$, the traveling salesman game has a core allocation for arbitrary edge lengths. We have shown above that this game can have an empty core if G has 7 nodes, $n = 6$, and all edges have unit length. Suppose that G has 6 nodes. If it does not contain a minor isomorphic to the graph in Figure 4 the core of its respective game is nonempty. Hence, assume further that G has such a minor. We were unable to show that for such a graph the core is nonempty. However, if all edges have unit lengths the core of this 6 node graph is nonempty since it contains the graph of Figure 4 as a subgraph, and hence contains a hamiltonian cycle.

We also note that the series parallel graph of Figure 3 constitutes a unique minimal example among all series parallel graphs. Suppose that G is series parallel, has at most 8 nodes, $n \leqslant 7$, and it does not contain the graph in Figure 3 as a minor. Then G can not contain the graphs in Figures 2 and 4 as minors. Therefore it follows from the above that the traveling salesman game associated with such a graph has a nonempty core.

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