On the Core of Cost Allocation Games Defined on Location Problems

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The literature on location theory has dealt mainly with finding "optimal" locations for centers which are to serve given demand points. No attention was given to the problem of allocating the costs of establishing the centers among the users, i.e., the demand points. In this paper we address this issue relating it to game theory concepts.

INTRODUCTION

The literature on location theory has dealt mainly with finding "optimal" locations for centers which are to serve given demand points. No attention was given to the problem of allocating the costs of establishing the centers among the users, i.e., the demand points. In this paper we address this issue relating it to game theory concepts.

Suppose that each user has a certain demand requirement. To meet these demands the users will have to set centers and, of course, absorb the incurred costs. Basically, it is possible for each group of users, a "coalition," to cooperate and establish centers satisfying the demand of its own members. The nature of the problem is such that if two disjoint coalitions unite, their total cost will not increase. Thus, viewing only the total cost for all users, there exists an incentive for them to act as a grand coalition. The question then arises as to the allocation of the total cost. Is there an allocation such that no group of users will have the incentive to split from the grand coalition and act on its own? In game theory terminology such an allocation is called a core allocation. Namely, a core allocation is such that no coalition can pay less than its part in this allocation if it establishes centers to meet the demands of its own members.

The existence of a core allocation becomes important since it seems to be a natural necessary condition for acceptability by the users.

This work discusses a class of location problems for which there always exists a core allocation. The core itself is characterized by a dual linear program to the location problem.

The Location Model

The following is one of the most elementary models used in studying the location of centers on networks. (The reader is referred to the vast literature on location theory, e.g. [7, 9, 14], where motivation is provided.)

Let $G = (N, E)$ be an undirected graph with N and E as its sets of nodes and edges respectively. Each edge is associated with a positive number called the length of the edge. Given two nodes $x, y \in N$, $d(x, y)$ is the length of a shortest (with respect to the sum of edge lengths) path on G connecting x and y. Two subsets D and C of N, are given. $D = \{p_1, \ldots, p_m\}$ is the set of demand points or users, while $C = \{q_1, \ldots, q_k\}$ denotes the set of nodes where a service center can be established. The cost of establishing a center at q_j is $w_j \geqslant 0$.

We ignore here capacity constraints on the centers, and assume that each center can serve any number of arriving users. The demand constraints are formulated as follows. User p_i demands that a center will be set at a distance of at most $r_i \geqslant 0$ from him. Assuming that all the demand constraints are to be met, the location problem is to find the minimum cost needed for setting centers fulfilling all the demands.

It is known that the above problem is *NP*-hard even when G is planar with all edges having length 1 and $w_j = 1$; $j = 1, \ldots, k$, $r_i = 1$; $i = 1, \ldots, m$.[8] However, polynomially bounded algorithms for solving the above model and some variants on a tree graph are available.[2, 9, 11, 13]

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THE COST ALLOCATION GAME

LET D be the set of players (users) and define a coalition to be a nonempty subset of D. A cooperative game on D is characterized by a real function V defined on the set of coalitions. (To simplify the notation we let D be the index set $\{1, \ldots, m\}$, and we identify a coalition S by a subset of $\{1, \ldots, m\}$.)

Given the location problem described above, define the cost allocation game on D as follows. If S is a set of users define $V(S)$ as the minimum cost required for setting centers meeting the demands of the users in S. In particular, $V(D)$ is the solution value of the above location problem. To ensure that $V(S)$ is finite we assume that the location problem is feasible, i.e. all demand constraints are met if a center is established at each point of C.

The core of V is defined as

$$\text{Core}(V) = \{x \in R^m | \Sigma_{i \in D} x_i = V(D),$$
$$\Sigma_{i \in S} x_i \leqslant V(S) \text{ for every coalition } S\}.$$

If $x \in \text{Core}(V)$ it is called a core allocation. x_i, then, denotes that part of the total cost paid by user p_i. The above characteristic function V is certainly monotone, i.e., for every pair of coalitions S_1 and S_2 with $S_1 \subseteq S_2$, $V(S_1) \leqslant V(S_2)$. Therefore, if x is a core allocation then $x \geqslant 0$. It is clear that no coalition of users would do better by breaking the cooperation between all users if a core allocation is used to split the total cost.

Thus, a core allocation possesses a desired stability property which seems to be necessary for an allocation to be acceptable by the users.

The following example illustrates that the core may be empty.

Example 1. Consider the graph C_4, the simple cycle having 4 nodes. Suppose that each one of the 4 edges has length 1. Furthermore, let both D and C be equal to N, the node set of C_4, with $r_i = w_i = 1$ for all nodes p_i. Then it is easily verified that the core is empty since the symmetric allocation is not in the core and $V(D) = 2$.

Observing that cycles may yield an empty core, let us turn to trees, i.e., graphs containing no cycles. We show that the core is nonempty for tree graphs and characterize its extreme points.

We start by a linear program formulation of the core of a general graph, requiring only the explicit computation of $V(D)$. Given a graph G and the above location model define the matrix $A = (a_{ij})$ by

$$a_{ij} = \begin{cases} 1 & \text{if } d(p_i, q_j) \leqslant r_i \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

LEMMA 2. *Given a graph G and its respective cost allocation game V, then*

$$\text{Core}(V) = \left\{x \in R^m | \Sigma_{i \in D} x_i = V(D) \text{ and } (A^T x)_j \leqslant w_j, j = 1, \ldots, k, x \geqslant 0\right\}.$$

Proof. Let $x \in \text{Core}(V)$. Then $x \geqslant 0$ and $\Sigma_{i \in D} x_i = V(D)$. Given j, $j = 1, \ldots, k$ let $S_j = \{i | a_{ij} = 1\}$, i.e., S_j is the set of users that can be served by a center located at q_j. Thus,

$$(A^T x)_j = \Sigma_{i \in S_j} x_i \leqslant V(S_j) \leqslant w_j.$$

Conversely, given $x \geqslant 0$ satisfying $(A^T x)_j \leqslant w_j, 1, \ldots, k$, we show that $\Sigma_{i \in S} x_i \leqslant V(S)$ for each coalition S.

Given S, suppose without loss of generality that $V(S)$ is achieved by setting centers at $q_1, q_2, \ldots, q_t$, i.e. $V(S) = \Sigma_{j=1}^t w_j$. Then it follows that $\Sigma_{j=1}^t a_{ij} \geqslant 1$ for each $i \in S$. Thus,

$$\Sigma_{i \in S} x_i \leqslant \Sigma_{i \in S}\left(\Sigma_{j=1}^t a_{ij}\right) x_i \leqslant \Sigma_{j=1}^t (A^T x)_j \leqslant \Sigma_{j=1}^t w_j = V(S). \quad \blacksquare$$

COROLLARY 3. *Given a graph G and its respective cost allocation game V, suppose that $V(D)$ is (explicitly) known. Then, if w_j is integral for all j, there exists a strongly polynomial algorithm for finding a core allocation or verifying that none exists. Also, for any $x \in R^m$ there exists a strongly polynomial scheme to test whether x is a core allocation.*

Proof. Follows directly from the above lemma and TARDOS' recent algorithm solving linear programs.[15] ■

As mentioned above, determining $V(D)$ for general graphs is NP-hard, and hence there is no general practical implication to the above corollary. For tree graphs $V(D)$ is polynomially computable.[9, 11] We next demonstrate that the core is nonempty for tree graphs.

THEOREM 4. *Suppose that G is a tree, with the respective cost allocation game V. If the location problem is feasible, i.e. $V(D) < \infty$, then the core is the set of optimal dual variables to the linear program*

$$\min\left\{\Sigma_{j=1}^k w_j y_j | (Ay)_i \geqslant 1, i = 1, \ldots, m; y \geqslant 0\right\}. \tag{2}$$

Proof. Let y_j be a 0–1 variable, taking on the value 1 if and only if a center is established at q_j. Then,

$$V(D) = \min\left\{\Sigma_{j=1}^k w_j y_j | (Ay)_i \geqslant 1, i = 1, \ldots, m, y \geqslant 0, y_j \in \{0, 1\}, j = 1, \ldots, k\right\}.$$

It is proved in [4, 11] that if G is a tree then all the extreme points of the polyhedron $\{y | (Ay)_i \geq 1, i = 1, \ldots, m;\ y \geq 0\}$ are 0–1 vectors. Hence, $V(D)$ is the minimum value of the linear program (2). Using the duality theorem of linear programming and Lemma 2, we obtain that the core consists of the set of optimal dual variables to the linear program,

$$\min\left\{\sum_{j=1}^{k} w_j y_j | (Ay)_i \geq 1,\ i = 1, \ldots, m,\ y \geq 0\right\}. \quad \blacksquare$$

It is shown in [4, 11] that when G is a tree, the matrix A is balanced i.e., it has no square submatrix of odd size with row and column sums equal to two. Thus, from [3], if w_j, $j = 1, \ldots, k$, is integral, then the linear program $\max\{\sum_{i=1}^{m} x_i | x \geq 0, (A^T x)_j \leq w_j, j = 1, \ldots, k\}$ has an integer solution vector x^*. Furthermore, if $w_j = 1$, $j = 1, \ldots, k$, then all extreme points of $\{x | x \geq 0, (A^T x)_j \leq 1, j = 1, \ldots, k\}$ are integer. Combining the above with Theorem 4 yields the following corollary.

COROLLARY 5. *Suppose that G is a tree and that the respective location problem is feasible.*

a. *If w_j, $j = 1, \ldots, k$ is integral there exists a core allocation with integral components.*
b. *If $w_j = a$, $j = 1, \ldots, k$ for some positive a, then every extreme point of the core has coordinates 0 or a. Moreover, the positive coordinates correspond to a subset of D of maximum cardinality such that no pair of its nodes can be served by one center.*

It is worth mentioning the connection between Theorem 4 and the results on linear programming games studied by OWEN.[10] Due to the integrality results used in the proof of Theorem 4 we note that the cost allocation game on a tree graph is in fact a linear programming game. As such its core is nonempty since it contains the set of optimal dual solutions.[10] It is well known that for general linear programming games the optimal dual set might be a proper subset of the core. AUBIN[1] has introduced the concept of the fuzzy core, and proved that it actually coincides with the optimal dual set. Lemma 2 and Theorem 4 exhibit a class of linear programming games for which the core itself coincides with the set of optimal dual solutions.

FURTHER EXTENSIONS

IN THE above model the only requirement of user p_i, $i = 1, \ldots, m$, was that a service center would be established within a distance of at most r_i from him. No reference was made to reliability issues. Suppose now that each service center can be in exactly one of two states: failure or functioning. The probability of being in a given state, say failure, is α for each center and is independent of the number, states and locations of the other centers. Originally, each user p_i has his own requirement for the probability that at least one service center, within a radius of r_i from him, would function. With the above probability distribution the requirement of p_i, $i = 1, \ldots, m$, can be translated and stated as follows: at least $a_i \geq 1$ centers should be established within a distance of at most r_i from p_i.

Again, define $V(S)$ as the minimum cost needed for setting centers satisfying the demand constraints of the coalition S. The next example shows that the core of this extended model may be empty even for tree graphs.

Example 6. Consider the tree $T = (N, E)$ where $N = \{p_1, p_2, p_3, p_4\}$ and $E = \{(p_1, p_2), (p_1, p_3), (p_1, p_4)\}$. Let $D = C = \{p_1, p_2, p_3, p_4\}$. Suppose that all edges have length 1 and $r_i = 1$, for all i. Also let $(a_1, a_2, a_3, a_4) = (w_1, w_2, w_3, w_4) = (2, 1, 1, 1)$. We then have

$$V(S) = \begin{cases} 3 & \text{if } S = N \\ 2 & \text{if } 2 \leq |S| \leq 3 \text{ or } S = \{p_1\} \\ 1 & \text{if } |S| = 1 \text{ and } S \neq \{p_1\}. \end{cases}$$

Using the symmetric roles played by p_2, p_3, and p_4 it is easily verified that the core is empty.

Next we prove that if all setup costs, w_j, $j = 1, \ldots, k$, are identical, then the core of the extended model is nonempty, provided the graph is a tree. Without loss of generality we now set $w_j = 1$, $j = 1, \ldots, k$.

We will consider a generalized model by introducing the following exogeneous constraint. Suppose that for each site q_j, $j = 1, \ldots, k$, there is an integer upper bound $b_j \geq 1$ on the total number of centers that can be established at q_j. (The setup cost per center is still one unit.) This constraint is applicable to every coalition of users, S, regardless of its size. It should be emphasized that the individual users are regarded as atoms that cannot be split further into "subusers" to form "subcoalitions." In particular, the requirement of user p_i for a_i centers cannot be partitioned into a_i subusers each requiring one center, to avoid the effect of the exogenous constraint. Formally we define $V(S)$, the minimum cost needed for setting centers satisfying the requirements of the users in S, by

$$\begin{aligned} V(S) = \min & \textstyle\sum_{j=1}^{k} y_j \\ \text{s.t. } & (Ay)_i \geq a_i, \quad i \in S \\ & b_j \geq y_j \geq 0, \quad j = 1, \ldots, k \\ & y_j \text{ is an integer}, \quad j = 1, \ldots, k. \quad (3) \end{aligned}$$

In spite of the presence of the exogeneous constraint the above generalized model still provides the users with the incentive to cooperate.

LEMMA 7. *The set function $V(S)$, defined in* (3), *is subadditive.*

Proof. Let S_1 and S_2 be two coalitions of users. Suppose that $V(S_1) = \Sigma_{j=1}^{k} y_j^1$, and $V(S_2) = \Sigma_{j=1}^{k} y_j^2$, where y^1 and y^2 are optimal solutions to (3) with $S = S_1$ and $S = S_2$, respectively. Define $z = (z_1, \ldots, z_k)$ by

$$z_j = \max\left(y_j^1, y_j^2\right), \qquad j = 1, \ldots, k.$$

Then, it is easily verified that z is feasible for (3) with $S = S_1 \cup S_2$. Therefore,

$$V(S_1 \cup S_2) \leqslant \Sigma_{j=1}^{k} z_j \leqslant \Sigma_{j=1}^{k} y_j^1 + \Sigma_{j=1}^{k} y_j^2 = V(S_1) + V(S_2). \qquad \blacksquare$$

We will now prove constructively that the core of the generalized game is nonempty.

It is proved in [4, 11] that if G is a tree then A is balanced. Therefore, if the integrality constraints in (3) are relaxed, there is still an optimal integer solution to the relaxed linear program.[3] Namely,

$$\begin{aligned} V(D) = \min \Sigma_{j=1}^{k} y_j & \\ \text{s.t. } (Ay)_i \geqslant a_i, & \quad i = 1, \ldots, m, \\ b_j \geqslant y_j \geqslant 0, & \quad j = 1, \ldots, k. \end{aligned} \tag{4}$$

The dual of (4) is

$$\begin{aligned} \max\left(\Sigma_{i=1}^{m} a_i z_i - \Sigma_{j=1}^{k} b_j u_j\right) & \\ \text{s.t. } (A^T z)_j - u_j \leqslant 1, & \quad j = 1, \ldots, k, \\ z \geqslant 0, \quad u \geqslant 0. & \end{aligned} \tag{5}$$

LEMMA 8. *Let G be a tree and suppose that (z^*, u^*) is an optimal extreme point solution to* (5). *Then z^* is a 0–1 vector, and u^* is integral.*

Proof. The positivity of b_j implies that $u_j^* = 0$ or else $(A^T z^*)_j - u_j^* = 1$. Applying the balancedness of A, (follows from [4, 11]), and Lemma 2.1 of [3] the proof is complete. $\blacksquare$

LEMMA 9. *Let (z^*, u^*) be an optimal extreme point solution to* (5).

a. *If $u_j^* = 0$ there exists at most one index i such that $a_{ij} z_i^* = 1$.*
b. *If $u_j^* > 0$ there exist exactly $u_j^* + 1$ distinct indices i such that $a_{ij} z_i^* = 1$.*
c. *If $z_i^* = 1$, then*

$$a_i \geqslant \Sigma_{\{j | a_{ij} u_j^* > 0\}} b_j. \tag{6}$$

Proof. (a) and (b) follow directly from (5) and the fact that z^* is a 0–1 vector. (c) is implied by the optimality of (z^*, u^*). If (6) does not hold for some i the optimality is contradicted by decreasing z_i^* and each u_j^*, satisfying $a_{ij} u_j^* > 0$, by 1. $\blacksquare$

The next theorem establishes the nonemptiness of the core constructively.

THEOREM 10. *Let G be a tree, and suppose that the setup costs for the centers, w_j, $j = 1, \ldots, k$, are identical. If the generalized model is feasible, i.e. all demands are satisfiable, then the core of the cost allocation game is nonempty.*

Proof. Without loss of generality we may suppose that $w_j = 1$, $j = 1, \ldots, k$. Let (z^*, u^*) be an optimal extreme point solution to (5). From Lemma 8 z^* is a 0–1 vector. Let $J = \{j | u_j^* \geqslant 1\}$. For each $j \in J$ choose $i(j)$ to be some index satisfying $a_{i(j), j} z_{i(j)}^* = 1$. Define

$$x^* = \Sigma_{i=1}^{m} a_i z_i^* e_i - \Sigma_{j \in J} \Sigma_{i \neq i(j)} a_{ij} z_i^* b_j e_i, \tag{7}$$

where e_i is the ith unit vector of dimension m.

It is claimed that x^* is a core allocation. Lemma 8(c) ensures that $x^* \geqslant 0$. Also,

$$\Sigma_{i=1}^{m} x_i^* = \Sigma_{i=1}^{m} a_i z_i^* - \Sigma_{j \in J} b_j u_j^* = V(D),$$

where the second equality follows from the duality of (4) and (5). It remains to show that $\Sigma_{i \in S} x_i^* \leqslant V(S)$ for each coalition $S \subseteq \{1, \ldots, m\}$.

Given a coalition S define $S^* = S \cap \{i | z_i^* = 1\}$. Then $V(S) \geqslant V(S^*)$ and $\Sigma_{i \in S} x_i^* = \Sigma_{i \in S^*} x_i^*$. Thus it suffices to prove that $V(S^*) \geqslant \Sigma_{i \in S^*} x_i^*$. Define $J_1 = \{j | \Sigma_{i \in S^*} a_{ij} z_i^* > 1\}$, $I_1 = \{i | i \in S^*$, and $a_{ij} z_i^* = 1$ for some $j \in J_1\}$, $\bar{I}_1 = S^* - I_1$. The fact that z^* is a 0–1 vector implies that if a center is within the radius, r_i, of some demand point p_i, $i \in \bar{I}_1$, then it is not within the radius of any other member of S^*. Thus, $\Sigma_{i \in \bar{I}_1} a_i$ centers are needed to meet the demands of the members in $\bar{I}_1$, and these centers do not contribute to meeting the demands of the members in I_1.

Suppose that to achieve $V(S^*)$ exactly $k_j \leqslant b_j$ centers are set at point q_j, $j \in J_1$. To satisfy a demand of a point p_i, $i \in I_1$, which is not met by the above $\{k_j, j \in J_1\}$ centers, we need $\max(0, a_i - \Sigma_{j \in J_1} a_{ij} k_j) = t_i$ additional centers. From the definition of J_1, these t_i centers will not contribute to meeting the demand of any member of I_1 but p_i itself.

Summarizing, we obtain

$$V(S^*) = \Sigma_{i \in \bar{I}_1} a_i + \Sigma_{j \in J_1} k_j + \Sigma_{i \in I_1} t_i$$
$$\geqslant \Sigma_{i \in \bar{I}_1} a_i + \Sigma_{j \in J_1} k_j$$
$$+ \Sigma_{i \in I_1}(a_i - \Sigma_{j \in J_1} a_{ij} k_j)$$
$$= \Sigma_{i \in S^*} a_i - \Sigma_{j \in J_1} k_j \big((\Sigma_{i \in I_1} a_{ij}) - 1\big)$$
$$\geqslant \Sigma_{i \in S^*} a_i - \Sigma_{j \in J_1} b_j \big((\Sigma_{i \in I_1} a_{ij}) - 1\big).$$

Using the definition of x^*, recall that for each $j \in J_1$ the quantity b_j is subtracted from the payments of all p_i, $i \in I_1$, (but possibly one), with $a_{ij} z_i^* = 1$. Thus,

$$V(S^*) \geqslant \Sigma_{i \in S^*} a_i - \Sigma_{j \in J_1} b_j \big((\Sigma_{i \in I_1} a_{ij}) - 1\big)$$
$$\geqslant \Sigma_{i \in S^*} x_i^*,$$

and the proof is complete. ∎

The above proof provides a scheme, (7), to generate core allocations for the generalized cost allocation game. The next example shows that this scheme might not exhaust all core allocations. For comparison purposes recall that Theorem 4 fully characterizes the core of the regular game as the set of optimal dual variables, (2).

Example 11. Consider the tree $T = (N, E)$ where $N = \{p_1, p_2, p_3\}$ and $E = \{(p_1, p_2), (p_2, p_3)\}$. Let $D = C = \{p_1, p_2, p_3\}$. Suppose that all edges have length 1 and $r_i = 1$, for $i = 1, 2, 3$. Also let $a_i = 2$, for $i = 1, 2, 3$, $(b_1, b_2, b_3) = (2, 1, 2)$, and $(w_1, w_2, w_3) = (1, 1, 1)$. Thus, the characteristic function of the generalized game is given by

$$V(\{i\}) = 2, \quad i = 1, 2, 3,$$
$$V(\{1,2\}) = V(\{2,3\}) = 2,$$
$$V(\{1,3\}) = V(\{1,2,3\}) = 3.$$

Therefore,

$$\text{Core}(V) = \{(x_2, x_2, x_3) | x_1 + x_2 + x_3 = 3,$$
$$x_1 + x_2 \leqslant 2,\ x_2 + x_3 \leqslant 2,$$
$$x_i \geqslant 0,\ i = 1, 2, 3\}.$$

The core has the following three extreme points:

$$(1,1,1), \quad (2,0,1) \quad \text{and} \quad (1,0,2).$$

The dual problem (5), which is used in (7) to construct the core allocations in Theorem 10, takes the following form

$$\max(2z_1 + 2z_2 + 2z_3 - 2u_1 - u_2 - 2u_3)$$
$$\text{s.t. } z_1 + z_2 \qquad -u_1 \leqslant 1$$
$$z_1 + z_2 + z_3 \qquad -u_2 \leqslant 1$$
$$z_2 + z_3 \qquad -u_3 \leqslant 1$$
$$z \geqslant 0, \quad u \geqslant 0.$$

The unique optimal solution is given by $(z_1^*, z_2^*, z_3^*, u_1^*, u_2^*, u_3^*) = (1, 0, 1, 0, 1, 0)$. Thus, the scheme given by (7) will only generate the extreme core allocations $(1, 0, 2)$ and $(2, 0, 1)$.

Remark 12. The core allocations constructed in (7) for the generalized game are integral. Similar integrality results for the core of the regular game are stated in Corollary 5. These properties are quite significant whenever the setup costs, the characteristic function V, as well as the allocations of the total cost, $V(D)$, can be expressed only in terms of some indivisible commodity (monetary unit), e.g., skilled workers, oil barrels, gold bullions, etc. The above integrality results guarantee that there is a stable (core) allocation even with that indivisibility restriction.

CONCLUDING REMARKS

THE ORIGINAL version of this work was written in 1980 and presented at the International Symposium on Locational Decisions, Skodsborg, Denmark, 1981. Since then the main results of this paper (Theorem 4, Corollary 5 and Theorem 10) on the existence of core allocations arising from location models have not been improved or generalized significantly.

From the proofs given above we note that the existence of core allocations on tree graphs follows from the balancedness property satisfied by the matrix A, defined by (1), which appears in the characterization of the core in Lemma 2. Such a matrix is in the class of totally balanced matrices which has been studied extensively since 1980. We refer the reader to [9] and the references cited there, for a detailed discussion on algorithms and structural results related to this class. In particular, we observe that core allocations of the types discussed in Corollary 5 and Theorem 10 above can now be computed in $0(n^2)$ time for a tree graph with n nodes.

The framework of the proof of Theorem 4 is directly applicable to other games whose respective optimization models are defined by a totally balanced matrix. One such model is the simple uncapacitated plant location problem on tree graphs. The cost allocation game defined by this problem is briefly discussed in [9]. Another example is the classical production lot size model of WANGER and WHITIN.[16] This model can be formulated as an instance of the uncapacitated plant location on a bitree. (A bitree is obtained from a tree by replacing each arc by a pair of oppositely directed arcs. These arcs are allowed to have different lengths). The total balancedness property is preserved for matrices A, defined by (1), when the underlying graph is a bitree.[12] Thus, when we look at the cost allocation game defined by the production lot size model in [16], viewing the demand points as the players, the same proof as in Theorem 4, will exhibit constructively that there exist core allocations.

With the exception of the above results the only other works we know of which focus on cost allocation schemes in location models on networks, and employ the cooperative game framework are [5, 6]. The model in [5] is different from the game discussed above in several respects and therefore the results are not comparable. Both are confined to tree networks. The location model in [5] considers only the single facility case; however, its assumptions on the utility functions of the users are less restrictive than those (implicitly) made in our model.

The cost allocation game in [6] is based on a connectivity location model. There are no proximity constraints on the distances between the users and the centers. The objective is to minimize the total cost of establishing the centers and connecting the users to them. Each user must be connected to some center, not necessarily the closest to him. It is shown in [6] that core allocations for this model exist when the underlying network is a tree.

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