Mathematical Programming 3 (1972) 390–391. North-Holland Publishing Company

SHORT COMMUNICATION

ON "AN EFFICIENT ALGORITHM FOR MINIMIZING BARRIER AND PENALTY FUNCTIONS"

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Received 17 June 1972

In [1], Lasdon presents a new algorithm which generates search directions by linearization of the objective and constraints about the current interior point, substituting these linearizations into the penalty function and minimizing the result.

In order to prove that the algorithm produces usable search directions (see [1, Theorem 3]) it is assumed (see [1, Assumption 3]) that the barrier function is differentiable, strictly convex and monotone decreasing. The lengthy proof is established by invoking the theory of subgradients and directional derivatives of convex functions. We show here a short and fundamental proof of the theorem while assuming only the differentiability and convexity of the barrier function. It is also observed that the same proof can be used in the exterior penalty function case (see [1, Theorem 10]).

Theorem. If (a) $\nabla P(x^*, r) \neq 0$, (b) *the barrier function B is differentiable and convex for all $z > 0$,* (c) *s^* solves* DF(x^*), *then*

$$\nabla P^t(x^*, r)\, s^* < 0\,.$$

Proof. Assume, on the contrary, that $\nabla P^t(x^*, r)\, s^* \geqq 0$, i.e.,

$$\left(b_0^{\mathrm{t}} + r \sum_{i=1}^{m} B'(a_i)\, b_i^{\mathrm{t}}\right) s^* \geq 0\,. \tag{1}$$

For all $i = 1, ..., m$, $B(a_i + b_i^{\mathrm{t}} s)$ is a differentiable convex function of s in the convex region D defined by $D = \{s\colon a_i + b_i^{\mathrm{t}} s > 0,\ i = 1, ..., m\}$.

This, together with the fact that $s = 0 \in D$ implies that

$$B(a_i + b_i^t s) \geq B(a_i) + B'(a_i)\, b_i^t s\,, \quad s \in D\,, \qquad i = 1, ..., m\,. \tag{2}$$

Since $s = 0$ is a feasible solution for DF(x^*) and s^* is optimal,

$$a_0 + b_0^t s^* + r \sum_{i=1}^{m} B(a_i + b_i^t s^*) < a_0 + r \sum_{i=1}^{m} B(a_i)\,. \tag{3}$$

Note that since $\nabla P(x^*, r) \neq 0$, strict inequality holds by [1, Theorem 2]. Using (2) and then (1), we get

$$a_0 + b_0^t s^* + r \sum_{i=1}^{m} B(a_i + b_i^t s^*) \geq a_0 + r \sum_{i=1}^{m} B(a_i) + b_0^t s^* +$$

$$+ r \sum_{i=1}^{m} B'(a_i) b_i^t s^* \geq a_0 + r \sum_{i=1}^{m} B(a_i)$$

which contradicts (3), and the theorem is proved.

Finally, note that [1, Theorem 10], the analogue of Theorem 3 to the exterior penalty function case, can be established by the above proof where B is replaced by any convex differentiable penalty function, e.g., $p(Z) = (\min(0, Z))^2$.

Reference

[1] L. Lasdon, "An efficient algorithm for minimizing barrier and penalty functions", *Mathematical Programming* 2 (1972) 65–106.