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On the core of network synthesis games

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We use polynomial formulations to show that several rational and discrete network synthesis games, including the minimum cost spanning tree game, satisfy the assumptions of Owen's linear production game model. We also discuss computational issues related to finding and recognizing core points for these classes of games.

Key words: Network synthesis, cooperative games, linear production games, core of a game.

1. Introduction

An n-person cooperative game in characteristic function form is a pair $(N; \bar{c})$, where $N=\{1,2,\ldots,n\}$ represents the set of players and $\bar{c}$ is the characteristic function. $\bar{c}$ is a function from the coalitions (subsets of N) to the real numbers. We are concerned with the linear production game introduced by Owen [24]. In this game each of n players is given a resource vector $b^i=(b_1^i, b_2^i, \ldots, b_m^i)$ $(i=1,2,\ldots,n)$. The m resources in themselves have no value, but they can be used to produce goods which can then be sold at a given market price. The production model is assumed to be linear, and a unit of the jth good $(j=1,\ldots,p)$, requires a_{kj} units of the kth resource $(k=1,\ldots,m)$, and can be sold at a price c_j. Another basic assumption of the production game is the following additivity assumption: any coalition S, $S\subseteq N=\{1,2,\ldots,n\}$ possesses a total of

$$b_k(S)=\sum_{i\in S} b_k^i$$

of the kth resource. Utilizing all of its resources the coalition S can realize a maximum profit $\bar{c}(S)$ given by

$$\begin{aligned}\bar{c}(S)=\max\ &\sum_{j=1}^{p} c_jx_j\\ \text{s.t.}\ &\sum_{j=1}^{p} a_{kj}x_j\leqslant b_k(S),\quad k=1,\ldots,m,\\ &x_j\geqslant 0,\quad j=1,\ldots,p.\end{aligned}\tag{1}$$

The linear production game is the pair $(N; \bar{c})$ where the characteristic function $\bar{c}$ is defined by (1). Owen proved that the core of this game is nonempty, and that a vector in the core can be easily computed from any optimal dual solution to (1) with $S=N$.

The above linear production game has been used as a unifying model to prove and explain the non-emptiness of the core of many cooperative games defined by various optimization problems, [18, 19, 27, 28]. An outstanding exception is the minimum cost spanning tree game which has been studied extensively in [1, 3, 4, 15, 16, 22, 23]. Several proofs of the non-emptiness of the core of this game were given. However, none of these proofs showed that this game could be classified within Owen's framework. Motivated by this class of games, Granot [12], introduced a generalized linear production game model that does not require the additivity assumption made by Owen. Using Edmonds' [5] characterization of the spanning tree problem polytope, Granot then demonstrated that the generalized model unifies the above class of games as well. Finally in that paper Granot applied his generalized model to prove the non-emptiness of the core of two other classes of network design cooperative games, which had not been previously studied. The proofs in [12] are based on exponential formulations, (e.g., Edmonds [5]), of the related optimization problems, and therefore they do not, in general, provide an efficient scheme to compute a vector in the core.

In Sections 2 and 3 we use a different formulation for a unifying network design cooperative game to exhibit that all the classes of games presented in [12] and [14] (including the minimum cost spanning tree game) do satisfy the assumptions of Owen's linear production game. Moreover, since the suggested formulation is polynomial in size a vector in the core can be computed efficiently by solving a single linear program.

In general, given a formulation for the linear production game, the set of core vectors generated by Owen's scheme from the dual solutions can be a proper subset of the core of the game. In fact, it was shown by Chvátal [2] that testing whether a given vector is not in the core of a linear production game which involves only one activity is already NP-hard. Chvátal's game can be viewed as a special case of our unifying network design cooperative game.

In view of this complexity result we focus in Section 4 on special classes. We consider the network synthesis games studied in [14], and provide an efficient (polynomial) characterization of the cores of these games and a strongly polynomial procedure for verifying whether a given vector is in the core of these games.

2. The continuous network synthesis game

Let $G=(N, E)$ be a directed network with $N=\{1, 2, \ldots, n\}$ and E denoting the node set and arc set respectively. We will consider an n-person cooperative game $(N; \bar{c})$, where N, the node set of G, represents the set of players and $\bar{c}$, the characteristic function, is obtained from some network design minimization model defined on G, e.g., a minimum spanning tree model or a minimum traveling salesman tour. For each coalition $S \subseteq N$, $\bar{c}(S)$ will be defined as the solution value of the minimization model reduced to the nodes in S.

Given a game $(N; \bar{c})$ in characteristic function form we focus on its core, which is one of the most intuitive and appealing concepts used in cooperative cost allocation models. Formally, the core of $(N; \bar{c})$ is defined as the set of all real vectors $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$, such that

$$\sum_{i \in S} x_i \leqslant \bar{c}(S) \quad \text{for all } S \subseteq N, \qquad \sum_{i \in N} x_i = \bar{c}(N).$$

(For convention purposes $\sum_{i \in \emptyset} x_i = \bar{c}(\emptyset) = 0$. Also note that, unlike Owen's model where the characteristic function is defined in terms of a maximization model, we deal with network minimization models.)

We now define a (continuous) network design cost allocation game, $(N; \bar{c})$ on the network G. Each ordered pair of distinct nodes $[k, l]$, $k, l \in N$, is associated with a commodity, say the $[k, l]$ commodity. Let $r_{kl} \geqslant 0$ denote the number of units of this commodity that must flow along the arcs of G from k to l. Suppose that initially all arc capacities are zero, so that no positive flow can be sent in the network. Furthermore, suppose that each arc $(i, j) \in E$ (the arc is directed from i to j), is associated with a capacity cost coefficient $c_{ij} \geqslant 0$, i.e., the cost of increasing the capacity of arc (i, j) is linear with slope c_{ij}. The network design (synthesis) problem is to find the arc capacities $\{y_{ij}\}$, $(i, j) \in E$, of minimum total cost that will permit the flow requirements $\{r_{kl}\}$ between the nodes. Naturally, two extreme cases of the problem come to mind, depending on whether the flows must occur simultaneously or not. We refer to them as the simultaneous case and the non-simultaneous case respectively. We will consider a generalized version that unifies both extreme cases.

The characteristic cost function $\bar{c}$ is defined as follows: For each coalition S, $\bar{c}(S)$ is the minimum cost of satisfying the flow requirements from S to N. Hence $\bar{c}(S)$ is the solution value of the reduced design problem obtained by setting $r_{kl} = 0$ for each $k \in N - S$ and $l \in N$. $\bar{c}(S)$ does not incorporate the cost of sending the flows from $N - S$ to S. This supposition is appropriate for a network in which the nodes represent transmitters (exporters) only. On the other hand, if a node plays a dual role and is both a transmitter and a receiver (exporter and importer), then the above cost term should definitely be included to yield a modified characteristic cost function, say $\bar{\bar{c}}$. However, since the core of the game $(N; \bar{c})$ is contained in the core of $(N; \bar{\bar{c}})$, $(\bar{c}(S) \leqslant \bar{\bar{c}}(S)$ for $S \subseteq N$, and $\bar{c}(N) = \bar{\bar{c}}(N))$, we will restrict ourselves to the game $(N; \bar{c})$ in proving the nonemptiness of both cores.

Granot [12] studied the non-simultaneous case of the game $(N; \bar{\bar{c}})$ with the additional assumption that the network G is undirected. He proved that the core is nonempty by using the exponential formulation of the related linear program, as appeared in [10, 11]. (We have modified the formulation to the directed case.)

$$\begin{aligned} \min \quad & \sum_{(i,j) \in E} c_{ij} y_{ij} \\ \text{s.t.} \quad & \sum_{\substack{(i,j) \in E \\ i \in X, j \in \bar{X}}} y_{ij} \geqslant r_{kl} \quad \text{for all } k, l \in N, k \neq l, \text{ and all cuts } (X, \bar{X}) \text{ separating nodes } k \text{ and } l, \\ & y_{ij} \geqslant 0 \quad \text{for all } (i, j) \in E. \end{aligned} \tag{2}$$

The above formulation does not satisfy the additivity assumption of Owen's model. However, Granot demonstrated that it constitutes a special case of the generalized linear production game in [12], thus implying the non-emptiness of the core. A shortcoming of his proof is that unlike Owen's model it does not provide an efficient scheme to compute a point in the core.

To obtain a formulation suitable for Owen's model, we use an additional set of variables, $\{f_{ij}^{kl}\}$, $(i, j) \in E$, $k, l \in N$, $k \neq l$. f_{ij}^{kl} will indicate the flow of the $[k, l]$ commodity along the directed arc (i, j). As above, the variables $\{y_{ij}\}$, $(i, j) \in E$, are the arc capacities. We also introduce constraints to accomodate for several possible interactions between the flows and the capacities.

To summarize, $\bar{c}$, the characteristic cost function of the network synthesis game, is given by

$$\bar{c}(S) = \min \sum_{(i,j)\in E} c_{ij} y_{ij}$$

$$\text{s.t.} \quad \sum_{\{j|(i,j)\in E\}} f_{ij}^{kl} - \sum_{\{j|(j,i)\in E\}} f_{ji}^{kl} \tag{3}$$

$$= \begin{cases} r_{kl} & \text{if } i = k, \\ 0 & \text{if } i \neq k, l, \end{cases} \quad \text{for all } k \in S, l \in N, k \neq l,$$

$$Af + By \leqslant 0,$$

$$f \geqslant 0, \quad y \geqslant 0,$$

f and y are the vectors with components $\{f_{ij}^{kl}\}$ and $\{y_{ij}\}$ respectively. A and B are data matrices of the appropriate dimensions.

To motivate the constraints $Af + By \leqslant 0$ consider the two extreme cases mentioned above. In the non-simultaneous case these constraints take the form

$$f_{ij}^{kl} \leqslant y_{ij} \quad \text{for } k, l \in N, k \neq l, (i, j) \in E.$$

The simultaneous case yields the constraints

$$\sum_{\substack{k,l\in N \\ k\neq l}} f_{ij}^{kl} \leqslant y_{ij} \quad \text{for } (i, j) \in E.$$

It is a simple matter to verify that the above cost allocation game satisfies the additivity assumption of the linear production game model of Owen [24]. Therefore, we can conclude that the core of the cost allocation game is nonempty. Each optimal dual solution to (3) with $S = N$ can be used to construct a core point for the game $(N; \bar{c})$ through a simple summation formula provided in [24].

We note that flow variables have been used before to model the Steiner Tree problem and some of its variants, e.g., Wong [30]. Our contribution here is in observing that flow variables might lead to formulations of the network design cost allocation game that are conformal with Owen's model. In the next section the potential of this formulation becomes even more apparent and significant when we discuss several discrete network design games.

A comment is in order regarding the version of the above model when the underlying graph G is undirected. This version can easily be converted to the directed case. For example, in the non-simultaneous case each undirected edge (i, j) is replaced by two oppositely directed arcs connecting i and j, and having the same cost coefficient $\frac{1}{2}c_{ij}$. Two variables, say y_{ij} and y_{ji}, are associated with this pair of arcs. We add to the formulation (3) the constraint $y_{ij} = y_{ji}$. With this transformation we conclude that the above results hold for the undirected case as well.

3. Discrete network design games

Viewing the above linear models it is natural to investigate the discrete versions as well. The latter models capture connectivity related problems. For example, the minimum spanning tree problem and the minimum Steiner graph problem can be formulated as special cases of the discrete model (3) corresponding to the non-simultaneous case. (Formally by the discrete model we refer to the case where the variables $\{y_{ij}\}$ are restricted to integral values.)

Consider the (directed) minimum spanning tree game model. In this model the objective is to find in G a minimum cost directed spanning tree rooted at some source node, say 1. For each coalition $S \subseteq N$, $\bar{c}(S)$ is a minimum cost (directed) tree that provides directed paths from the nodes in S to the source.

Using Edmonds' [5] characterization of the directed spanning tree polytope in a generalized linear production game, Granot [12] proved that the cost allocation game induced by this model has a nonempty core. (Earlier results concerning the more restricted undirected case appeared in [1, 3, 4, 15, 16, 22, 23].)

We shall now use the formulation (3) to show that even this (discrete) spanning tree game model satisfies the additivity assumption of Owen's model. In particular, the non-emptiness of the core will follow directly from [24].

The characteristic cost function of the directed spanning tree game $(N; \bar{c})$ is given by

$$\bar{c}(S) = \min \sum_{(i,j)\in E} c_{ij} y_{ij}$$

$$\text{s.t.} \quad \sum_{\{j \mid (i,j)\in E\}} f_{ij}^{k1} - \sum_{\{j \mid (j,i)\in E\}} f_{ji}^{k1} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k, 1, \end{cases} \quad \text{for all } k \neq 1, k \in S, \tag{4}$$

$$f_{ij}^{k1} \leqslant y_{ij},$$

$$f_{ij}^{k1} \geqslant 0,$$

$$y_{ij} \in \{0, 1\}, \text{ for all, } k \neq 1, \text{ and all } (i,j) \in E.$$

For each coalition (subset) $S \subseteq N$, let $\bar{c}^*(S)$ denote the optimal value of the linear programming relaxation of the above discrete program, i.e., the solution value of the program obtained by omitting the binary constraints on the variables $\{y_{ij}\}$, $(i, j) \in E$. From the discussion in Section 2 the cost allocation game with $\bar{c}^*$ as its characteristic function has a nonempty core. The claim is that each point of the core of the game $(N; \bar{c}^*)$ is also a core point of the discrete game $(N; \bar{c})$. Indeed, we trivially have $\bar{c}^*(S) \leq \bar{c}(S)$ for each subset $S \subseteq N$, and the claim holds since $\bar{c}(N) = \bar{c}^*(N)$, as follows from Edmonds [6] and Fulkerson [8] and was shown explicitly by Wong [30].

It follows from the above and Owen's model [24], that a core point to the minimum cost directed spanning tree allocation game can easily be computed from an optimal dual solution to the linear programming relaxation obtained from (4) for $S = N$. We note in passing that one of those dual solutions is the particular core point given in the original proof of the non-emptiness of the core in [15], i.e., if T is a minimum cost directed spanning tree then x_i, the cost allocated to player i, is the cost of the (unique) arc of T leaving i. In the undirected case, Granot and Huberman [16] have provided graph theoretic methods to generate some points in the core. Some of their results are valid for the directed case as well. The general problem of characterizing the extreme points of the core of the minimum spanning tree game remains open.

In the above game the interest is to provide connectivity to a given source from *all* other nodes, i.e., with the exception of the source each node of the graph is a player. If we consider a model where the set of players is a proper subset of $N - \{1\}$, then the resulting game may have an empty core. The example in Figure 1 demonstrates this point. Suppose that all edges there have a cost of one unit. If node 1

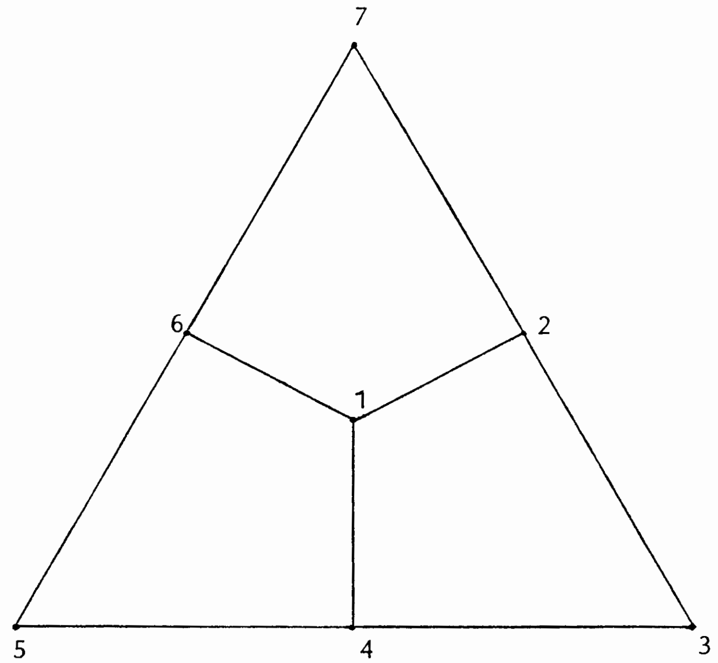

Fig. 1.

there is chosen as the source, and the set of players consists only of nodes 3, 5 and 7, then the core of the respective cost allocation game is empty. In fact this example is a minimal one from the following respect. First note that in our model a game with two players has a core allocation. Thus, at least three players are needed for a game with an empty core. Moreover, it is shown in [13] that the core of the game is nonempty if the underlying graph is series-parallel. Clearly the graph in Figure 1 is not of this type. However, if we delete any edge from it we obtain a series-parallel graph. A 4-player game demonstrating the possible emptiness of the core of the above model follows from Megiddo [22].

The minimum spanning tree game model discussed above is only one example of a discrete game for which Owen's linear model provides an efficient scheme to generate a core point from a dual solution. Another example is the class of flow games considered by Kalai and Zemel [18, 19]. (The discrete game there is defined by considering integer data, and letting the flows represent indivisible commodities.) The latter model generalizes the discrete assignment game introduced by Shapley and Shubik [27]. The reduction of the discrete game to Owen's linear model is based on the total unimodularity property of the node arc incidence matrix of a directed network.

A third example is the class of discrete location models on tree graphs considered in Tamir [28] and Kolen and Tamir [20]. In this case the reduction to the linear model follows from the balancedness property of the matrix used to define the optimization problem.

A fourth example is the Steiner tree problem game. It has been demonstrated above that in general this game can have an empty core. However, it was shown by D. Granot and F. Granot [13] that for series-parallel graphs this discrete game can be reduced to the generalized linear production game in [12]. In fact, we can easily reduce it to Owen's model by using the formulation in (3) and the polyhedral characterization of Steiner trees in series-parallel graphs in [26].

As a final example we refer to the traveling salesman game presented in [25]. In this game $\bar{c}(S)$ is defined as a minimum length (cost) tour starting at the source node, visiting each node in S at least once and returning to the source node. It is shown in [29] that the core of this discrete game might be empty even if the graph G has seven nodes. Sufficient conditions for a nonempty core are also provided in [29]. Again, the constructive proof is based on reducing the game to Owen's model by using (3).

4. Submodular games

As mentioned above some core points of a linear production game can easily be generated from the dual solutions for the underlying linear program. For some linear and discrete models (e.g. [20, 27, 28]) the core coincides with the set of dual solutions, but this is not true in general. In fact, for the latter case the problem of testing whether a given point is not in the core of a linear production game as defined in [24] is known to be NP-hard, see Chvátal [2]. In such a recognition problem we assume the input to consist of a given rational point and the linear model. The coefficients of the linear model are also rational. With this setup, for every coalition S the characteristic value, $\bar{c}(S)$, can be computed in polynomial time by solving a linear program. The linear production game used by Chvátal to demonstrate the above NP-hardness result has only one activity. This game can be reduced and viewed as a special case of our unifying network design cooperative game (3), for some simple tree graph. Thus, studying the solvability of the above recognition problem we will focus on the special classes of the network design games studied in [14], and show that their recognition problem can be resolved polynomially.

Consider a cost allocation cooperative game $(N; \bar{c})$ defined by Owen's model, and suppose that $\bar{c}$ is submodular, i.e.

$$\bar{c}(S_1 \cup S_2) + \bar{c}(S_1 \cap S_2) \leq \bar{c}(S_1) + \bar{c}(S_2) \quad \text{for all } S_1, S_2 \subseteq N. \tag{5}$$

A given point $x = (x_i)$, $i \in N$, is in the core if it satisfies

$$\sum_{i \in S} x_i \leq \bar{c}(S) \quad \text{for all } S \subseteq N \quad \text{and} \quad \sum_{i \in N} x_i = \bar{c}(N).$$

Therefore, x is a core point if and only if

$$\sum_{i \in N} x_i = \bar{c}(N)$$

and the minimum of the submodular function

$$\left\{ \bar{c}(S) - \sum_{i \in S} x_i \right\}, \quad S \subseteq N,$$

defined on the power set of N, is non-negative. Since in our case $\bar{c}(S)$ can be computed in polynomial time it follows from [17] that the above minimum can be computed in polynomial time.

The submodularity assumption also enables us to recognize efficiently whether a given point is the kernel or the nucleolus of the game. (It is known that for a submodular cost allocation game the kernel is a singleton and it coincides with the nucleolus [21].)

The kernel of the game $(N; \bar{c})$, with respect to the grand coalition, is the set of all points $x = (x_i)$, $i \in N$, such that

$$x_i \leq \bar{c}(\{i\}) \quad \text{for all } i \in N, \qquad \sum_{i \in N} x_i = \bar{c}(N),$$

and for all $\{p, q\} \subseteq N$, either

$$S_{pq}(N, x, \bar{c}) \leq S_{qp}(N, x, \bar{c}) \text{ or } x_q = \bar{c}(\{q\}).$$

For any pair of players p, q in N, $S_{pq}(N, x, \bar{c})$ is defined by

$$S_{pq}(N, x, \bar{c}) = \min \left\{ \bar{c}(S) - \sum_{i \in S} x_i : S \subseteq N, p \in S, q \notin S \right\}.$$

Given a point x one can test in polynomial time whether x is the kernel point of the game if the values $\{S_{pq}(N, x, \bar{c})\}$, $\{p, q\} \subseteq N$, can be computed efficiently. Given a pair of players p, q in N, we note that $S_{pq}(N, x, \bar{c})$ is the minimum value of a submodular function defined on the lattice $\{S\colon S \subseteq N, p \in S, q \notin S\}$. Again it follows from [17] that $S_{pq}(N, x, \bar{c})$ can be evaluated in polynomial time when $\bar{c}$ is submodular and defined by Owen's linear production game, [24].

Although we can now recognize whether a given point is the kernel (or nucleolus) of a submodular game the problem of finding the kernel point efficiently is still open.

The network synthesis games discussed above are not in general submodular. Two types of such games that possess the submodularity property are studied by Granot and Hojati [14]. However, no efficient procedure is given in [14] for testing membership in the cores of these games. We next provide compact representations for these cores that involve only a polynomial number of variables and constraints. In particular, it will easily follow from our formulation that testing membership in the core amounts to finding a minimum cut in some auxiliary networks defined by the respective games. In the two models the underlying graph $G=(N, E)$, $N=\{1, 2, \ldots, n\}$, is undirected, and the requirement matrix $\{r_{kl}\}$ is symmetric.

The first model discussed in [14] is the undirected simultaneous network design model defined above. It is shown in [14] that the characteristic cost function $\bar{c}(S)$, $S \subseteq N$, for this case is given by

$$\bar{c}(S) = \tfrac{1}{2}\left(\sum_{j \in S} \sum_{k \in N} r_{jk} D_{jk} + \sum_{j \in S} \sum_{k \in N-S} r_{jk} D_{jk} \right), \tag{6}$$

where D_{jk} denotes the length (cost) of a shortest (cheapest) path connecting nodes j and k in N.

Given a vector $x=(x_j)$, $j \in N$, consider the problem of testing its membership in the core. x should satisfy

$$\sum_{j \in S} x_j \leq \bar{c}(S) \quad \text{for all } S \subseteq N,$$

with equality holding for $S=N$. Thus, x is in the core if and only if

$$\bar{c}(N) = \sum_{j \in N} x_j \leq \sum_{j \in N-S} x_j + \tfrac{1}{2} \sum_{j \in S} \sum_{k \in N} r_{jk} D_{jk} + \tfrac{1}{2} \sum_{j \in S} \sum_{k \in N-S} r_{jk} D_{jk} \quad \text{for all } S \subseteq N. \tag{7}$$

For each $S \subseteq N$, let $h(S)$ denote the right-hand side of (7). Clearly, $h(S)$ is a submodular function. x is in the core if and only if $\min\{h(S)\colon S \subseteq N\} \geq \bar{c}(N)$. (In fact, since $h(N) = \bar{c}(N)$ this minimum should be equal to $\bar{c}(N)$.) We next demonstrate that $h(S)$ is a cut function defined on some auxiliary directed network. Let $G'=(N', E')$, be a directed graph with node set $N' = N \cup \{s, t\}$. E' is defined as follows. For each node j in N there exist a directed arc connecting the source s to j with capacity x_j, and a directed arc connecting j to the sink t with capacity

$$\tfrac{1}{2} \sum_{k \in N} r_{jk} D_{jk}.$$

For each pair of distinct nodes $j, k \in N$, there are two arcs, oppositely directed, connecting j and k. Both arcs have the same capacity bound equal to $\frac{1}{2}r_{jk}D_{jk}$.

Each coalition $S \subseteq N$ corresponds to an s–t cut on G'. Moreover, the value of such a cut is exactly $h(S)$. Therefore, x is in the core of the game if and only if the minimum s–t cut on G' is equal to

$$\bar{c}(N) = \tfrac{1}{2} \sum_{j \in N} \sum_{k \in N} r_{jk} D_{jk}.$$

Using the duality between the minimum cut and the maximum flow problems, we can characterize the core by the following compact polynomial formulation.

For each arc $(i, j) \in E'$, let f_{ij} denote the flow on the arc. Then x is in the core if and only if

$$\sum_{j \in N} x_j = \bar{c}(N)$$

and the following flow problem is feasible.

$$\sum_{\{j \mid (i,j) \in E'\}} f_{ij} - \sum_{\{j \mid (j,i) \in E'\}} f_{ji} = \begin{cases} \frac{1}{2} \sum_{j \in N} \sum_{k \in N} r_{jk} D_{jk}, & i = s, \\ 0, & i \neq t, s, \end{cases}$$
$$0 \leqslant f_{ij} \leqslant \begin{cases} x_j & \text{if } i = s, \\ \frac{1}{2} \sum_{k \in N} r_{ik} D_{ik} & \text{if } j = t, \\ \frac{1}{2} r_{ij} D_{ij} & \text{if } i \neq s \text{ and } j \neq t. \end{cases} \tag{8}$$

The above "network flow" formulation provides a compact polynomial characterization of the core. It is appropriate to mention that it was shown in [14] that the vector $x = (x_1, x_2, \ldots, x_n)$, where

$$x_j = \tfrac{1}{2} \sum_{k \in N} r_{jk} D_{jk},$$

is both the Shapley value and the nucleolus of this game.

The second submodular network design game considered in [14], is a special case of the undirected non-simultaneous game defined above. The additional assumption here is that the underlying undirected graph $G = (N, E)$ is a complete graph with equal capacity cost coefficients. This common cost coefficient is assumed to be equal to one unit. It is shown in [14] that the characteristic cost function $\bar{c}(S)$, $S \subseteq N$, is submodular and is given by

$$\bar{c}(S) = \tfrac{1}{2}\left(\sum_{j \in S} \max\{r_{jk} : k \in N\} + \sum_{j \in N-S} \max\{r_{jk} : k \in S\} \right). \tag{9}$$

Therefore $x = (x_j)$, $j \in N$, is in the core if and only if

$$\bar{c}(N) = \sum_{j \in N} x_j \leqslant \sum_{j \in N-S} x_j + \tfrac{1}{2} \sum_{j \in S} \max\{r_{jk} : k \in N\}$$
$$+ \tfrac{1}{2} \sum_{j \in N-S} \max\{r_{jk} : k \in S\} \quad \text{for all } S \subseteq N.$$

(In particular, as shown in [14], the vector $x = (x_j)$, where $x_j = \frac{1}{2} \max\{r_{jk} : k \in N\}$ is always in the core of this game.) Equivalently, x is a core point if and only if

$$\bar{c}(N) \leq \sum_{j \in S} x_j + \tfrac{1}{2} \sum_{j \in N-S} \max\{r_{jk} : k \in N\}$$
$$+ \tfrac{1}{2} \sum_{j \in S} \max\{r_{jk} : k \in N - S\} \quad \text{for all } S \subseteq N. \tag{10}$$

For each $S \subseteq N$, let $g(S)$ denote the right-hand side of (10). ($g(N) = \sum_{j \in N} x_j$ and $g(\emptyset) = \bar{c}(N)$.) x is in the core if and only if $\min\{g(S)\colon S \subseteq N\} \geq \bar{c}(N)$. (Again, since $g(N) = \bar{c}(N)$ the equality should hold.) We will now show that this minimum is equal to the minimum cut of the following auxiliary directed graph $G'' = (N'', E'')$.

To understand the formal definition of G'' the reader should note that N'' contains a node for each node in $N = \{1, 2, \ldots, n\}$, and instead of connecting a node i in N directly to all other nodes in N, (as was done for the previous model), we use intermediate auxiliary nodes. Thus, in G'', each node $i \in N$, will be associated with an additional set of nodes N_i, $|N_i| = n - 1$. As before, s and t are the source and sink of G'' respectively. Formally, $N'' = N \cup N_1 \cup \cdots \cup N_n \cup \{s, t\}$, where for each $i \in N$, $N_i = \{[i, j]\colon j \in N, j \neq i\}$.

We next construct the arcs in E''. First, for each node $i \in N$ there exist a directed arc connecting s to i with capacity $\frac{1}{2}\max\{r_{ik} : k \in N\}$, and a directed arc connecting i to the sink t with capacity x_i. To define the arcs that are incident to N_i, we sort the elements $\{r_{i,j}\}$, $j \in N$, $j \neq i$, and suppose that $r_{i,j(1)} \geq r_{i,j(2)} \geq \cdots \geq r_{i,j(n-1)}$. Then for each $p = 1, \ldots, n-2$, we establish a directed arc from $[i, j(p)]$ in N_i to $j(p)$ in N with infinite capacity, and a directed arc from $[i, j(p)]$ to $[i, j(p+1)]$ with capacity $\frac{1}{2} r_{i,j(p+1)}$. We also set a directed arc from i to $[i, j(1)]$ with capacity $\frac{1}{2} r_{i,j(1)}$, and an arc from $[i, j(n-1)]$ to $j(n-1)$ in N with infinite capacity. (See Figure 2.)

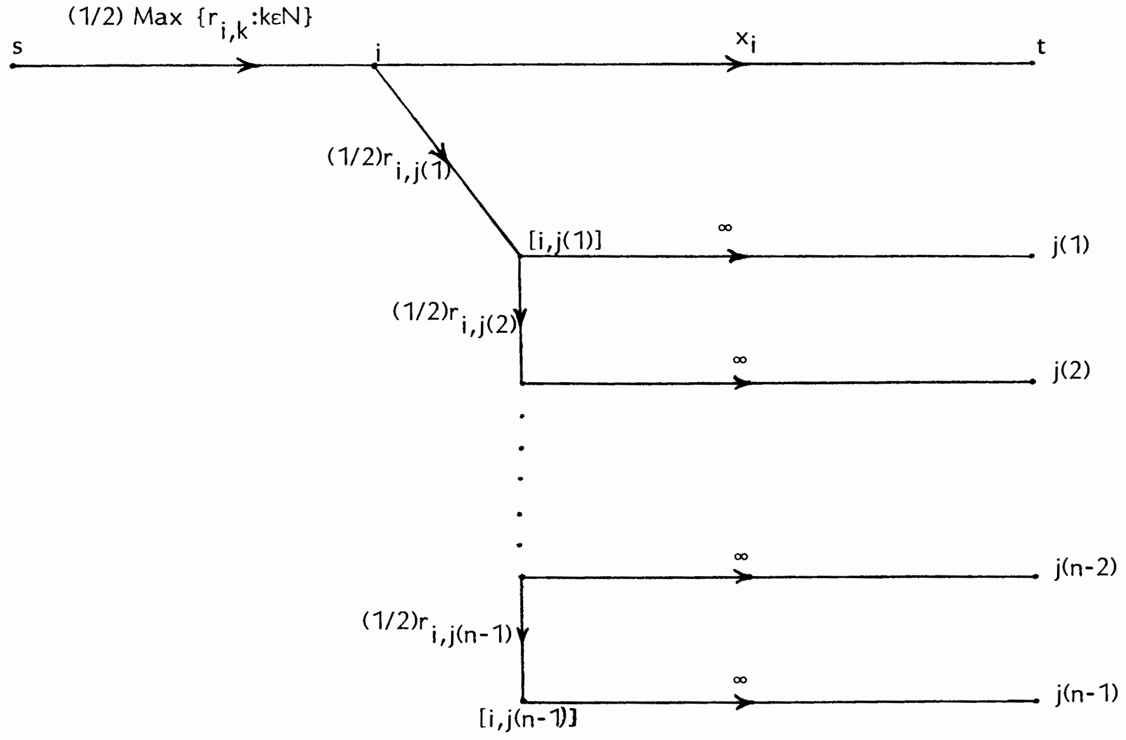

Fig. 2.

The next result proves the equivalence between the minimum cut problem on G'' and the membership in the core.

Lemma. *The minimum s–t cut on* $G''=(N'', E'')$ *is equal to* $\min\{g(S)\colon S\subseteq N\}$ *where* $g(S)$ *is defined as the right-hand side of* (10).

Proof. Suppose first that $g(S^*)=\min\{g(S)\colon S\subseteq N\}$ for some $S^*\subseteq N$. We define an s–t cut in G'' whose value is equal to $g(S^*)$.

Each s–t cut is a partition of N'' into two subsets containing s and t respectively. We characterize the cut with value $g(S^*)$ by specifying the set $\bar{S}$ of the partition that contains s. Suppose first that S^* is either empty or equal to N. If S^* is empty set $\bar{S}=\{s\}$, otherwise set $\bar{S}=N''-\{t\}$. In both cases the value of the cut is $g(S^*)$.

Suppose next that S^* is a nonempty proper subset of N. For each $i\in S^*$, let p_i be the smallest of all indices p, $1\leqslant p\leqslant n-1$, such that there exists a node $[i,j(p)]$ in N_i with $j(p)\in N-S^*$. In particular, $r_{i,j(p_i)}=\max\{r_{ij}: j\in N-S^*\}$.

Define $\bar{S}=S^*\cup\{s\}\cup\{[i,j(p_i-1)]\colon i\in S^*\}$. (For convenience $[i,j(0)]\equiv i$.) It is now easy to verify that the capacity value of this cut is indeed $g(S^*)$.

Next consider a minimum s–t cut in G''. Let $\bar{S}\subseteq N''$ be the side of the cut containing s. Let $S^*=N\cap\bar{S}$. We will show that the value of the cut is at least $g(S^*)$. First, we note that if S^* is either empty or equal to N the value of the cut $\bar{S}$ is certainly greater than or equal to $g(S^*)$. Thus suppose that S^* is a proper nonempty subset of N.

Consider $i\in S^*$, and let p_i be defined as above, i.e., $r_{i,j(p_i)}=\max\{r_{ij}: j\in N-S^*\}$. The capacity of the arc connecting $[i,j(p_i)]$ to $j(p_i)$ is infinite. Since $\bar{S}$ is an optimal cut it follows that $[i,j(p_i)]$ is not in $\bar{S}$. The node $i\equiv[i,j(0)]$ is in $\bar{S}$. Therefore there exists $\bar{p}$, $0\leqslant\bar{p}<p_i$, such that $[i,j(\bar{p})]$ is in $\bar{S}$ and $[i,j(\bar{p}+1)]$ is not in $\bar{S}$. The contribution of this part of the network to the capacity value of the cut is at least $\frac{1}{2}r_{i,j(\bar{p})}$, where $r_{i,j(\bar{p})}\geqslant r_{i,j(p_i)}=\max\{r_{ij}: j\in N-S^*\}$.

Thus, the total value of the cut defined by $\bar{S}$ is at least

$$\sum_{j\in S^*} x_j+\tfrac{1}{2}\sum_{j\in N-S^*}\max\{r_{jk}: k\in N\}+\tfrac{1}{2}\sum_{j\in S^*}\max\{r_{jk}: k\in N-S^*\}=g(S^*).$$

This completes the proof of the Lemma. □

As in the previous model, one can now introduce flow variables for the network $G''=(N'', E'')$ and obtain a compact and polynomial characterization for the core of this model which is analogous to (8).

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