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LEAST MAJORIZED ELEMENTS AND GENERALIZED POLYMATROIDS

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We prove that a bounded generalized polymatroid has a least weakly submajorized (supermajorized) vector. Such a vector simultaneously minimizes every nondecreasing (nonincreasing), symmetric and quasi-convex function defined on the generalized polymatroid. The same result holds also for the set of integer vectors of a bounded integral generalized polymatroid. We then extend these results to more general sets, and discuss several computational aspects.

1. Introduction. Veinott (1971) has shown that for any feasible linear network flow model, there is a flow which simultaneously minimizes every symmetric and quasi-convex function of the flows emanating from a single distinguished node called the source. (The symmetry concept introduced by Veinott is weaker than the conventional definition which requires the function to be invariant under permutations of its arguments.) Important applications of this result to deterministic production-distribution models, inventory redistribution models and several maximum likelihood estimation problems are discussed in Veinott (1971). Using the results by Dutta and Ray (1989) we can also claim that the core of a convex game, which is known to be nonempty (Shapley 1971), possesses a vector which simultaneously minimizes every symmetric and convex function defined on the core. Related results are also discussed by Megiddo (1974, 1977) and Fujishige (1980). We unify and extend the above results to generalized polymatroids, which have been introduced by Hassin (1978, 1982) and Frank (1984). We then extend these results to more general polyhedra, and discuss related computational aspects.

2. Notation and basic definitions. For any $x = (x_1, \dots, x_n)$ in R^n, let $x_{[1]} \geqslant \cdots \geqslant x_{[n]}$ denote the components of x in decreasing order, and let $x_\downarrow = (x_{[1]}, \dots, x_{[n]})$ denote the *decreasing rearrangement of* x. Similarly, let $x_{(1)} \leqslant \cdots \leqslant x_{(n)}$ denote the components of x in increasing order, and let $x_\uparrow = (x_{(1)}, \dots, x_{(n)})$ denote the *increasing rearrangement of* x. For any $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in R^n we say that x is *lexicographically less than or equal to* (*greater than or equal to*) y if $x = y$ or there exists an index t, $t = 1, \dots, n$, such that $x_t < y_t$ ($x_t > y_t$) and $x_i = y_i$ for any $i < t$. We say that x is *majorized* by y, $x \ll y$, if

$$\sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]}, \qquad k = 1, \dots, n-1,$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}.$$

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(If x is majorized by y we also say that y majorizes x.) We say that x is *weakly submajorized* by y, $x \ll_w y$, if

$$\sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]}, \qquad k = 1, \ldots, n.$$

Similarly, x is said to be *weakly supermajorized* by y, $x \ll^w y$, if

$$\sum_{i=1}^{k} x_{(i)} \geqslant \sum_{i=1}^{k} y_{(i)}, \qquad k = 1, \ldots, n.$$

It is easily verified that $x \ll y$ if and only if $x \ll_w y$ and $x \ll^w y$. We also note the following proposition which we need for future reference.

PROPOSITION 2.1. (1) *If $x \ll_w y$ then $x_\downarrow$ is lexicographically less than or equal to $y_\downarrow$.*

(2) *If $x \ll^w y$ then $x_\uparrow$ is lexicographically greater than or equal to $y_\uparrow$.*

The next results characterize the above terms of majorization (Marshall and Olkin 1979).

THEOREM 2.2. *The following conditions are equivalent.*

(1) $x \ll y$.

(2) $\sum_{i=1}^{n} f(x_i) \leqslant \sum_{i=1}^{n} f(y_i)$ *for all continuous convex functions* f.

(3) $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ *and* $\sum_{i=1}^{n} \max(0, x_i - a) \leqslant \sum_{i=1}^{n} \max(0, y_i - a)$ *for all real numbers* a.

THEOREM 2.3. *The following conditions are equivalent*:

(1) $x \ll_w y$.

(2) $\sum_{i=1}^{n} f(x_i) \leqslant \sum_{i=1}^{n} f(y_i)$ *for all continuous nondecreasing convex functions* f.

(3) $\sum_{i=1}^{n} \max(0, x_i - a) \leqslant \sum_{i=1}^{n} \max(0, y_i - a)$ *for all real numbers* a.

THEOREM 2.4. *The following conditions are equivalent*:

(1) $x \ll^w y$.

(2) $\sum_{i=1}^{n} f(x_i) \leqslant \sum_{i=1}^{n} f(y_i)$ *for all continuous nonincreasing convex functions* f.

(3) $\sum_{i=1}^{n} \max(0, a - x_i) \leqslant \sum_{i=1}^{n} \max(0, a - y_i)$ *for all real numbers* a.

Least majorized elements. Let S be a subset of R^n. A vector x in S is said to be a *least majorized element* of S if $x \ll y$ for all vectors y in S. A vector x in S is a *least weakly submajorized* (*supermajorized*) element of S if $x \ll_w y$ ($x \ll^w y$) for all vectors y in S. There are several results in the literature proving the existence of least majorized elements for certain subsets S. Veinott (1971) considers a linear flow problem in a directed network. Defining S to be the projection of the set of feasible flows on the subset of outflows from the source, he proves that S has a unique least majorized element. He also proves that S^*, the subset of integer points in S, has a least majorized element. (An integer least majorized element is not necessarily unique.) A similar flow problem is considered by Megiddo (1974), where it is shown that the respective set S constitutes a polymatroid. In his later paper (Megiddo 1977) Megiddo presents a strongly polynomial algorithm which finds the unique lexicographically maximum element x^* in S, i.e., $x^*_\uparrow$ is lexicographically larger than or equal to $y_\uparrow$ for all vectors y in S. (From Veinott's results and Proposition 2.1 it follows that x^* is in fact the unique least majorized element of S.) Special cases of the above models are discussed in Barel and Tamir (1981) and Granot et al. (1993). Fujishige (1980) extends the results of Megiddo to a general polymatroid and presents an algorithm to find a lexicographically optimal base of the polymatroid with respect to an arbitrary positive weight vector d. This weighted model is closely related to the concept of d-majorization introduced by Veinott (1971). Neither Megiddo nor Fujishige relate their results on lexicographically optimal bases to the stronger concept of majorization. (From Proposition 2.1 we note that if an arbitrary set has a least majorized element it is clearly lexicographically optimal. However, every convex and compact set S has a unique lexicographically maximum element, but might not have a least majorized element.) The fact that a polymatroid has a least majorized base is shown by Dutta and Ray (1989). They consider the core of a convex game as defined by Shapley (1971), which corresponds to a polymatroid. (Strictly speaking the former is defined as a contra-polymatroid; see next section.) We will extend and unify the above results by proving that a bounded generalized polymatroid contains both least submajorized and least supermajorized elements.

3. Generalized polymatroids and least majorized elements. Let E be a finite set and let b: $2^E \to R \cup \{+\infty\}$ be a set function defined on the power set of E, with $b(\phi) = 0$. We say that b is *submodular* if the submodular inequality

$$b(X) + b(Y) \geqslant b(X \cap Y) + b(X \cup Y)$$

holds for every subsets X and Y of E. A set function p is *supermodular* if $-p$ is submodular. The set functions, p and b are *compliant* if the following cross inequality

$$b(X) - p(Y) \geqslant b(X - Y) - p(Y - X)$$

holds for every subset X and Y of E. A pair (p, b) is called a *strong pair* if $-p$ and b are submodular and p and b are compliant. For a strong pair (p, b) the polyhedron

$$Q(p, b) = \{x \in R^E | p(A) \leqslant x(A) \leqslant b(A) \text{ for every } A \subseteq E\}$$

is called a *generalized polymatroid* (g-polymatroid). We use the common notation where $x(A) = \Sigma_{i \in A} x_i$.

If p and b are integer-valued, $Q(p, b)$ is called an *integral g-polymatroid.* Generalized polymatroids were introduced first by Hassin (1978, 1982) and independently by Frank (1984). The reader is referred to Bouchet (1987), Bouchet and Cunningham (1993), Chandrasekaran and Kabadi (1988), Dunstan and Welsh (1973), Frank and Tardos (1988), Fujishige (1984, 1991), Kabadi and Chandrasekaran (1990), Nakamura (1993), and Qi (1988), for structural and algorithmic results concerning g-polymatroids and related concepts. Important special cases are polymatroids, contra-polymatroids, base polyhedra and submodular polyhedra (Frank and Tardos 1988). If p is identically zero and b is submodular and monotone increasing, i.e., $b(X) \geqslant b(Y)$ if $X \supseteq Y$, then $Q(p, b)$ is a *polymatroid* (Edmonds 1970). If p is supermodular and b is identically $= +\infty$, then $Q(p, b)$ is a *contra-polymatroid* (Frank 1992, Frank and Tardos 1988). Note that the latter coincides with the core of a convex game (Shapley 1971). If b is submodular and p is identically $= -\infty$, $Q(p, b)$ is a *submodular polyhedron* (Fujishige 1991). Finally when b is submodular and p is defined by $p(A) = b(E) - b(E - A)$ for any subset A of E, $Q(p, b)$ is a *base polyhedron* (Fujishige 1991).

Our main result is that a bounded g-polymatroid $Q(p, b)$ contains both a least weakly submajorized element and a least weakly supermajorized element. Moreover, if $Q(p, b)$ is integral and $Q^*(p, b)$ is the set of all integer vectors in $Q(p, b)$, then

$Q^*(p, b)$ possesses a least weakly submajorized element as well as a least weakly supermajorized element.

Suppose first that $Q(p, b)$ is a bounded integral g-polymatroid. To simplify the notation we assume that $E = \{1, \dots, n\}$. Since majorization is not affected under an addition of the same constant to each component of every vector x, and since $Q(p, b)$ is bounded, we will assume without loss of generality that $p(\{i\})$ is nonnegative for each i in E. Next, consider the set

$$Q' = \left\{ x_{i,j},\ i = 1, \dots, n;\ j = 1, \dots, b(\{i\}) | 0 \leqslant x_{i,j} \leqslant 1 \right.$$

$$\left. \text{and } \sum_{j=1}^{b(\{i\})} x_{i,j} = x_i,\ i = 1, \dots, n \text{ for some } x \in Q(p, b) \right\}.$$

It follows from Frank and Tardos (1988) that Q' is a bounded integral g-polymatroid, i.e., there exists an integral valued strong pair (p', b') such that $Q' = Q(p', b')$.

THEOREM 3.1. *Let $Q(p, b)$ be a bounded integral g-polymatroid. Then $Q^*(p, b)$, the set of integer points in $Q(p, b)$, has a least weakly submajorized element.*

PROOF. From Theorem 2.3 it is sufficient to prove that there is an integer vector in $Q^*(p, b)$ that will simultaneously maximize any function of the form $\sum_{i=1}^{n} f(x_i)$ where f is nonincreasing and concave. We can assume without loss of generality that every such function f is also piecewise linear with integer breakpoints. Maximizing the above function is equivalent to maximizing the following function,

$$\sum_{i=1}^{n} \sum_{j=1}^{b(\{i\})} (f(j) - f(j-1)) x_{i,j},$$

over the set of integers points in Q'. Since the function f is nonincreasing and concave, it follows that all the coefficients of the above linear function are nonpositive, and $f(j) - f(j-1) \geqslant f(j+1) - f(j)$ for every positive integer j. In particular, there exists an ordering of the variables $\{x_{i,j}\}$ according to the size of their respective coefficients, and this ordering is independent of the specific function f. Using the greedy algorithm in Hassin (1978, 1982) we conclude that there is an integer vector $(x^*_{i,j})$ in Q' that simultaneously maximizes the above objective over Q' for all such functions f. Therefore, it follows that the vector x^*, defined by $x^*_i = \sum_{j=1}^{b(\{i\})} x^*_{i,j}$, $i = 1, \dots, n$, is a least weakly submajorized element in $Q^*(p, b)$, the set of integer vectors in $Q(p, b)$. □

Using similar arguments we can also conclude the following theorem.

THEOREM 3.2. *Let $Q(p, b)$ be a bounded integral g-polymatroid. Then $Q^*(p, b)$, the set of integer points in $Q(p, b)$, has a least weakly supermajorized element.*

To prove the continuous analogues of Theorems 3.1 and 3.2 and show that $Q(p, b)$ has both a (unique) least weakly submajorized element and a (unique) least weakly supermajorized element we use a similar approach. For the sake of brevity we mention only the main differences between the proofs. Consider, for example the existence of a least weakly submajorized element in $Q(p, b)$. In the integer case discussed above, we have proved that there is an element that simultaneously maximizes the function $\sum_{i=1}^{n} f(x_i)$ for all nonincreasing concave functions f, and transformed the maximization to the set of integer points in Q'. In the continuous case we use the result in Theorem 2.3 and look at the case where $f(x_i) = \min(0, a - x_i)$ instead. Again, we transform the maximization problem to Q' and apply the greedy algorithm in Hassin (1978, 1982) to conclude that $Q(p, b)$ has a least weakly submajorized element.

We summarize the above discussion with the following corollary.

COROLLARY 3.3. *If $Q(p, b)$ is a bounded g-polymatroid and $p(E) = b(E)$, then $Q(p, b)$ contains a unique least majorized element. Moreover, if $Q(p, b)$ is integral, then the set of integer points in $Q(p, b)$ has a least majorized element.*

4. **Extensions and final comments.** We note that least submajorized and least supermajorized elements of a bounded g-polymatroid can be computed in polynomial time by the ellipsoid methods in Grötschel et al. (1988). For example, assuming without loss of generality that $Q(p, b)$ is in R^n_+, and using Theorem 2.3, we note that the unique least submajorized element of a bounded g-polymatroid $Q(p, b)$, say x^*, is the unique minimizer of the quadratic

$$f(x) = \sum_{i=1}^{n} x_i^2$$

over $Q(p, b)$.

Also, from Proposition 2.1, $x^*_{\downarrow}$ is lexicographically smaller than or equal to $y_{\downarrow}$ for all vectors y in $Q(p, b)$.

x^* can be found in strongly polynomial time by modifying the procedure in Fujishige (1980) and Groenvelt (1991) which is applicable to polymatroids. The latter procedure can now be implemented to solve any convex separable quadratic over a polymatroid in strongly polynomial time since its complexity is dominated by the effort to minimize a (strongly) polynomial number of submodular functions. As shown in Grötschel et al. (1988), the minimization of a submodular function can be performed in strongly polynomial time.

The existential results in §3 hold for a class of bounded polyhedra (polytopes) which is larger than the class of bounded g-polymatroids. Let S be a nonempty polytope in R^n, and let c be a vector in R^n. Consider the maximization of the linear form $c'x$ over S. Suppose that S has the property that the greedy algorithm Dunstan and Welsh (1973), gives an optimal solution for every vector c satisfying $c \geqslant 0$ or $c \leqslant 0$. Combining our results in §3 with those in Dunstand and Welsh (1983, §5), we conclude that S has both a least weakly submajorized vector and a least weakly supermajorized vector. As an example of such polytopes consider the class of bounded delta-submodular polyhedra (Bouchet 1987, Bouchet and Cunningham 1993, Chandrasekaran and Kabadi 1988, Kabadi and Chandrasekaran 1990, Nakamura 1993, Qi 1988). It follows from Chandrasekaran and Kabadi (1988), Kabadi and Chandrasekaran (1990), and Nakamura (1993) that if S is such a polyhedron, then the greedy algorithm gives an optimal solution for every vector c in R^n. Also, note that if S is a bounded integral delta-submodular polyhedron, then S^*, the discrete set of integral points in S, has a least weakly submajorized element and a least weakly supermajorized element. We conjecture that this latter property holds also for bounded jump systems considered in Bouchet and Cunningham (1993).

As noted above Veinott (1971) has introduced a generalization of the majorization concept to positive weight vectors. Let d be a given positive vector in R^n. An $n \times n$ matrix P is called *d-stochastic* if $P \geqslant 0$, $\mathbf{1}'P = \mathbf{1}'$ and $Pd = d$. ($\mathbf{1}$ is the vector in R^n all of whose components are equal to 1.) If x and y are vectors in R^n we say that x is *d-majorized* by y if $x = Py$ for some d-stochastic matrix P. It is well known (Marshall and Olkin 1979) that when $d = \mathbf{1}$, $\mathbf{1}$-majorization coincides with the majorization definition given in §2 above.

Veinott (1971) proves that for any positive vector d the network flow polymatroid defined above contains a least d-majorized element. (See Galo et al. (1989) for a very efficient algorithm to find this element.) Using the above techniques we can generalize his results and conclude that it holds for arbitrary polymatroids. In fact, the least d-majorized base of a polymatroid is the unique lexicographically optimal base discussed by Fujishige (1980).

Finally, we note that unlike the case $d = \mathbf{1}$ (see Corollary 3.3), even the set of integer bases of a network flow polymatroid might not possess a least d-majorized element for some positive vector d. Consider the (flow) polymatroid defined by the set $\{(x_1, x_2) | x_1 \geqslant 0, x_2 \geqslant 0, x_1 \leqslant 1 \text{ and } x_1 + x_2 \leqslant 3\}$. Let $(d_1, d_2) = (1, 4)$. The unique least majorized base is $(\frac{3}{5}, \frac{12}{5})$. There are only two integral bases: $(1, 2)$ and $(0, 3)$. It is easily verified that none of them is d-majorized by the other.

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