POLYNOMIAL FORMULATIONS OF MIN-CUT PROBLEMS

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Abstract

We discuss formulations of several minimum cut problems on directed and undirected graphs, where the number of variables and constraints is polynomial.

Consider a directed graph $G = (V, A)$, where $V = \{v_1, \ldots, v_n\}$ is a finite set of nodes and A is a finite set of arcs. Let $m = |A|$. For every arc (v_i, v_j), we assume $v_i \neq v_j$, and call v_i its *tail* and v_j its *head*. We will also say that the arc is directed from v_i to v_j. (Assume that G has no parallel directed arcs, i.e. there is no pair of arcs with identical tails and identical heads.) Every arc $(v_i, v_j) \in A$ is associated with a nonnegative number, $c_{i,j}$, called the *capacity* of the arc. A *cut* is a partition of V into two nonempty sets $(S, V - S)$. The *capacity of the cut*, $C(S)$, is defined by

$$C(S) = \sum \{c_{i,j} : v_i \in S, \quad v_j \in V - S \quad \text{and} \quad (v_i, v_j) \in A\} \ .$$

Assume that two distinct nodes s and t are given, s being called the *source* and t the *sink*. An *$s - t$ cut* is a cut $(S, V - S)$ such that $s \in S$ and $t \in V - S$. The *minimum $s - t$ cut problem* is to find an $s - t$ cut with minimum capacity. More generally, given is a set D of p ordered pairs of nodes (terminals) in V,

$$D = \left\{ [v_{i(q)}, v_{j(q)}] | q = 1, \ldots, p, \right\} \ ,$$

$(i(q) \neq j(q), \ q = 1, \ldots, p)$. A (nonsimultaneous) *$D$-cut* is a cut $(S, V - S)$, such that $v_{i(q)} \in S$ and $v_{j(q)} \in V - S$ for some q, $q = 1, \ldots, p$. The *minimum D-cut problem* is to find a D-cut with minimum capacity. A D-cut of minimum capacity is called a *minimum D-cut.*

In this note we discuss polynomial formulations of the above minimum cut problems as linear systems.

Minimum $s - t$ Cuts.

Two classical linear formulations of the minimum $s - t$ cut problem are known (Ford and Fulkerson 1962). In the first formulation there is a variable u_i for each node $v_i \in V$ and a variable $y_{i,j}$ for each arc $(v_i, v_j) \in A$. (To simplify the notation suppose that $s = v_1$ and $t = v_n$. Also let $A' = \{(i, j) : (v_i, v_j) \in A\}$.)

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{(i,j)\in A'} c_{i,j} y_{i,j} \\
\text{s.t.} \quad & u_i - u_j + y_{i,j} \geq 0, \quad \forall (i,j) \in A' , \\
& u_n - u_1 = 1 \\
& y \geq 0 .
\end{aligned}
\tag{1}
$$

In the above formulation all the extreme points are binary vectors.

The second formulation involves only the arc variables $\{y_{i,j}\}$. However, the number of constraints can be exponential in $|V|$. Define a $[v_1, v_n]$-*path* as a sequence $a_1, a_2, \ldots, a_q$, of arcs in A such that v_1 is the tail of a_1, the tail of a_{i+1} is identical to the head of a_i, $i = 1, \ldots, q-1$, and v_n is the head of a_q. (Each path P is also viewed as a subset of A. Let $P' = \{(i,j) : (v_i, v_j) \in P\}$.)

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{(i,j)\in A'} c_{i,j} y_{i,j} \\
\text{s.t.} \quad & \sum_{(i,j)\in P'} y_{i,j} \geq 1, \quad \forall [v_1, v_n] - \text{path } P , \\
& y \geq 0 .
\end{aligned}
\tag{2}
$$

It is known (Ford and Fulkerson 1962), that all the extreme points in formulation (2) are binary vectors. This integrality property is preserved even when we restrict all the variables to be bounded by 1.

There is also another formulation of the $s-t$ cut problem which has only a polynomial number of constraints and uses only the arc variables. This formulation follows directly from (2).

Suppose, without loss of generality, that A contains all possible directed arcs. Consider the following linear program:

$$\begin{aligned}
\text{Minimize} \quad & \sum_{(i,j)\in A'} c_{i,j} y_{i,j} \\
\text{s.t.} \quad & y_{1,n} \geq 1 \; , && \text{(3a)} \\
y_{i,j} + y_{j,k} \geq y_{i,k}, \quad & \forall i,j,k = 1,\ldots,n, && \text{(3b)} \qquad (3) \\
& i \neq j, \quad i \neq k \text{ and } j \neq k \; , \\
& y \geq 0 \; .
\end{aligned}$$

Theorem 1. *The linear program (3) has a binary optimal solution, y^*. Moreover, y^* is an optimal solution to the minimum $v_1 - v_n$ cut problem.*

Proof: We start by proving that if y is a feasible solution to problem (3) then it is also feasible for problem (2). Consider a directed path P in G, connecting the source v_1 with the sink v_n. Let $y_{1,j(1)}, y_{j(1),j(2)}, \ldots, y_{j(k),n}$ be the sequence of variables corresponding to the sequence of arcs in P. To prove that the sum of these variables is greater than or equal to 1, we may assume that P is a simple path, i.e. $j(\ell) \neq 1, n$, for $\ell = 1, \ldots, k$, and $j(p) \neq j(q)$ for all $p, q = 1, \ldots, k$, $p \neq q$. To prove that y is feasible for problem (2) we use the metric constraints (3b) repeatedly, then (3a) to see that

$$y_{1,j(1)} + y_{j(1),j(2)} + \cdots + y_{j(k),n} \geq y_{1,n} \geq 1 \; .$$

Thus, we conclude that problem (3) is a restriction of problem (2). Let y' be an optimal binary solution of problem (2). To prove the theorem we will now show that there is a binary vector y^*, $y^* \leq y'$, and y^* satisfies the constraints (3a)-(3b). With each arc (v_i, v_j) associate the length $y'_{i,j}$. Next, define $y^*_{i,j}$ to be the length of a shortest path from v_i to v_j. It is clear that y^* is a binary vector satisfying $y^* \leq y'$. Moreover by construction y^* satisfies the metric inequalities (3b) as well as the constraint (3a). This completes the proof. □

We could not find an explicit exposition of formulation (3) in the literature. The case of an undirected graph is implicit in the "Japanese Theorem" on multicommodity flows, (see Lomonosov (1985)), which is attributed to Iri (1970/71) and Onaga and Kakusho (1971).

The homogeneous constraints (3b) are called the *metric constraints*, and they define the *metric cone*. The reader is referred to Grishukhin (1992) and Laurent and Poljak (1992) for recent results on the extreme rays of this cone in the case of an undirected graph.

If $n \geq 5$ the above feasible set contains noninteger extreme points. However, these noninteger extreme points are not optimal for the minimum $s-t$ cut problem when the objective vector is positive. Barahona and Mahjoub (1986) and Barahona (1993) have used the metric inequalities to characterize the cut polytope of an undirected graph which is not contractible to K_5, the complete graph on 5 nodes.

It follows from Theorem 1 that the linear program (3) has a binary optimal solution which also satisfies the constraints

$$y_{i,j} + y_{j,k} + y_{i,k} \leq 2 \ , \tag{3c}$$

$$y_{i,j} + y_{j,k} + y_{k,i} \leq 1, \quad \forall i,j,k = 1,\ldots,n, \quad i \neq j, \quad i \neq j \quad \text{and} \quad j \neq k \ . \tag{3d}$$

(Note that the constraints (3b)-(3c) imply that all variables are bounded above by 1.)

The dual of problem (3) suggests an interesting polynomial formulation for the classical maximum $s-t$ flow problem.

$$\begin{aligned}
&\text{Maximize} \quad F \\
&\quad \text{s.t.} \quad \sum_{\substack{k=1 \\ k\neq i,j}}^{n} (x_{i,j,k} + x_{k,i,j} - x_{i,k,j}) \leq c_{i,j}, \quad (i,j) \in A', \quad (i,j) \neq (1,n), \\
&\qquad\quad \sum_{k=2}^{n-1} (x_{1,n,k} + x_{k,1,n} - x_{1,k,n}) + F \leq c_{1,n}, \\
&\qquad\quad x_{i,j,k} \geq 0, \quad \forall i,j,k = 1,\ldots,n, \quad i \neq j, \quad i \neq k, \quad j \neq k, \\
&\qquad\quad F \geq 0.
\end{aligned} \tag{3*}$$

To interpret the above formulation, let $x_{i,j,k}$ be the value of the flow circulation through the ordered triplet (triangle), (i,j,k), i.e., the induced flow through the arcs $(v_i, v_j), (v_j, v_k)$ is $x_{i,j,k}$, while the flow through the arc (v_i, v_k) is $-x_{i,j,k}$. The variable F indicates a separate flow through the arc (v_1, v_n). Thus, the above constraints require that the total flow on each arc does not exceed its capacity.

Theorem 2. *Suppose that all arc capacities are integer. Then there is an optimal integer solution to (3*), x, F, which satisfies $x_{1,n,k} = x_{k,1,n} = 0$, $k = 2, \ldots, n-1$, and $F = \sum_{t=2}^{n-1} x_{1,t,n} + c_{1,n}$.*

Proof: From Theorem 1 we know that the optimal objective value of problem (3*) is equal to the minimum value of a $v_1 - v_n$ cut. Using the dual of formulation (2), the latter is also the maximum number of $[v_1, v_n]$ - paths that can be packed, without violating the arc capacities. Thus, it is sufficient to show how to convert a simple $[v_1, v_n]$ - path into a feasible solution to (3*), which satisfies the above properties. Without loss of generality consider a simple $[v_1, v_n]$ - path which has at least one intermediate node. Let $v_1, v_{i(1)}, \ldots, v_{i(\ell)}, v_n$, be the sequence of nodes on the path. Define a solution to problem (3*) by its nonzero components. Set

$$F = 1, \quad x_{1,i(1),i(2)} = x_{1,i(2),i(3)} = x_{1,i(3),i(4)} = \cdots$$
$$x_{1,i(\ell-1),i(\ell)} = x_{1,i(\ell),n} = 1.$$

It is easy to verify that this solution is indeed an integer feasible solution which satisfies the above properties.This completes the proof. □

The polynomial formulation (3) can be extended to any minimum multicommodity cut problem which can be formulated as a covering problem of a set of paths by the arcs (edges) of the graph. As an example consider the two-commodity flow/cut problem on an undirected graph $G = (V, E)$, (Hu 1963). Given are two pairs of terminals (s_1, t_1) and (s_2, t_2). A (simultaneous) *two-commodity cut* is a set of edges separating each one of the two pairs. Given a nonnegative capacity function on the edges of E, the problem is to find a two-commodity cut of minimum total capacity. Following Hu (1963), (see also Schrijver 1983, Section 7), the problem can be formulated as:

$$\begin{aligned} \text{Minimize} \quad & \sum_{\{i,j\} \in E'} c_{\{i,j\}} y_{\{i,j\}} \\ \text{s.t.} \quad & \sum_{\{i,j\} \in P'} y_{\{i,j\}} \geq 1, \quad \forall [s_p, t_p] - \text{path } P, \quad p = 1, 2\ , \\ & y \geq 0\ . \end{aligned} \tag{4}$$

For each edge $\{v_i, v_j\}$ in E, $c_{\{i,j\}}$ and $y_{\{i,j\}}$ denote the capacity and the variable of the edge, respectively. Assuming that G is a complete undirected graph with $V = \{v_1, \ldots, v_n\}$,

$E = \{\{v_i, v_j\} : i,j = 1,\dots,n,\ i \neq j\}$ and $E' = \{\{i,j\} : i,j = 1,\dots,n,\ i \neq j\}$, let $s_1 = v_1$, $s_2 = v_2$, $t_1 = v_n$, $t_2 = v_{n-1}$. Using the arguments as in the proof of Theorem 1 we conclude that problem (4) is equivalent to problem (5).

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{\{i,j\} \in E'} c_{\{i,j\}} y_{\{i,j\}} \\
\text{s.t.} \quad & y_{\{1,n\}} \geq 1 , \\
& y_{\{2,n-1\}} \geq 1 . \\
& y_{\{i,j\}} + y_{\{j,k\}} \geq y_{\{i,k\}}, \quad \forall i,j,k = 1,\dots,n, \quad i \neq j, \quad i \neq k, \quad j \neq k .
\end{aligned}
\tag{5}
$$

The reader is referred to Schrijver (1983, Section 7) for a survey of additional models that can be formulated as path covering problems.

Minimum D-Cuts.

In this section we show that the polynomial formulations (1) and (3) of the minimum $s-t$ cut problem can be extended to the minimum D- cut problem.

Consider the following linear program.

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{q=1}^{p} \sum_{(i,j) \in A'} c_{i,j} y^q_{i,j} \\
\text{s.t.} \quad & u^q_i - u^q_j + y^q_{i,j} \geq 0, \quad q = 1,\dots,p, \text{ and } (i,j) \in A', \\
& \sum_{q=1}^{p} (u^q_{j(q)} - u^q_{i(q)}) = 1, \\
& y^q_{i,j} \geq 0, \quad q = 1,\dots,p, \text{ and } (i,j) \in A'.
\end{aligned}
\tag{6}
$$

The dual of (6) is the following (nonsimultaneous) p-commodity flow problem.

$$
\begin{aligned}
\text{Maximize} \quad & F \\
\text{s.t.} \quad & \sum_{\{j|(i,j) \in A'\}} x^q_{i,j} - \sum_{\{j|(j,i) \in A'\}} x^q_{j,i} = \begin{cases} F & \text{if } i = i(q) \\ 0 & \text{if } i \neq i(q), j(q) \\ -F & \text{if } i = j(q) \end{cases} \\
& x^q_{i,j} \leq c_{i,j}, \quad q = 1,\dots,p, \quad \text{and} \quad (i,j) \in A', \\
& x^q_{i,j} \geq 0, \quad q = 1,\dots,p, \quad \text{and} \quad (i,j) \in A'.
\end{aligned}
\tag{7}
$$

(To interpret (7) as a p-commodity flow problem, for each $q = 1, \ldots, p$, and $(i,j) \in A'$, let $x^q_{i,j}$ denote the flow of commodity q in arc (v_i, v_j). The node $v_{i(q)}$ is the source of commodity q and $v_{j(q)}$ is its sink.)

We prove that problem (6) is a valid formulation of the minimum D-cut problem.

Theorem 3. *Suppose that the minimum D-cut is achieved by a cut $(S, V - S)$, where $v_{i(r)} \in S$ and $v_{j(r)} \in V - S$, for some r, $r = 1, \ldots, p$. Then, an optimal solution to problem (6) is defined by setting*

$$\begin{aligned}
&u^r_i = 0, \text{ for } v_i \in S, \quad u^r_i = 1, \text{ for } v_i \in V - S, \\
&u^q_i = 0, \ q = 1, \ldots, p, \ q \neq r, \text{ and } i = 1, \ldots, n, \text{ and} \\
&y^q_{i,j} = \max\{0, u^q_j - u^q_i\}, \ q = 1, \ldots, p \text{ and } (i,j) \in A'.
\end{aligned} \tag{8}$$

Proof: The vector defined above is certainly feasible for problem (6). By definition the objective value associated with it is the optimal objective value of the minimum $v_{i(r)} - v_{j(r)}$ cut problem. Let F_r denote the latter value. For each $q = 1, \ldots, p$, the value of the minimum $v_{i(q)} - v_{j(q)}$ cut is at least F_r. Therefore, by the Max flow-Min cut Theorem for a single commodity, there is a feasible solution to problem (7) with $F = F_r$. Thus, using the duality of problems (6) and (7), we conclude that the vector defined by (8) is an optimal solution to problem (6). □

The following result is also worth mentioning.

Theorem 4. *The coefficient matrix of problem (6) is totally unimodular.*

For the sake of brevity we skip the proof of Theorem 4, and note that it can be derived, for example, by the characterization in Ghouila-Houri (1962).

Next we show how to extend formulation (3) to the minimum D-cut problem.

We assume without loss of generality that the arc set A contains all possible directed arcs. Consider the following linear program.

$$\begin{aligned}
\text{Minimize} \quad & \sum_{q=1}^{p} \sum_{(i,j)\in A'} c_{i,j} y_{i,j}^{q} \\
\text{s.t} \quad & y_{i,j}^{q} + y_{j,k}^{q} \geq y_{i,k}^{q}, \quad \forall i,j,k = 1,\ldots,n, \\
& i \neq j, \quad i \neq k, \quad j \neq k, \quad \text{and} \quad q = 1,\ldots,p, \\
& \sum_{q=1}^{p} y_{i(q),j(q)}^{q} \geq 1, \\
& y \geq 0.
\end{aligned} \tag{9}$$

We will prove that (9) is a linear programming formulation of the minimum D-cut problem. We start by reformulating the maximum flow problem (7) as a maximum problem of packing $\left[v_{i(q)}, v_{j(q)}\right]$ - paths, $q = 1,\ldots,p$.

For each $q = 1,\ldots,p$, let $\{P_l^q\}$, $l \in L^q$, be the set of all simple $\left[v_{i(q)}^q, v_{j(q)}^q\right]$ - paths. Associate a variable x_l^q with the path P_l^q. Consider the following packing problem.

$$\begin{aligned}
\text{Maximize} \quad & F \\
\text{s.t.} \quad & \sum_{\{l \mid l\in L^q, (v_i,v_j)\in P_l^q\}} x_l^q \leq c_{i,j}, \; q = 1,\ldots,p, \text{ and } (i,j) \in A', \\
& \sum_{l\in L^q} x_l^q \geq F, \quad q = 1,\ldots,p, \\
& x_l^q \geq 0, \quad q = 1,\ldots,p, \quad \text{and} \quad l \in L^q, \quad \text{and} \\
& F \geq 0.
\end{aligned} \tag{10}$$

The dual of the packing problem (10) is given by,

$$\begin{aligned}
\text{Minimize} \quad & \sum_{q=1}^{p} \sum_{(i,j)\in A'} c_{i,j} y_{i,j}^{q} \\
\text{s.t.} \quad & \sum_{\{(i,j)\mid (v_i,v_j)\in P_l^q\}} y_{i,j}^{q} \geq z^q, \quad \forall \text{ path } P_l^q, l \in L^q, \\
& \sum_{q=1}^{p} z^q \geq 1, \\
& y_{i,j}^{q} \geq 0, \quad q = 1,\ldots,p, \quad \text{and} \quad (i,j) \in A', \\
& z^q \geq 0, \quad q = 1,\ldots,p.
\end{aligned} \tag{11}$$

Problems (10) and (11) correspond to problems (7) and (6) respectively. In particular, if the optimal solution to the minimum D-cut problem is attained by some cut $(S, V-S)$, where $v_{i(r)} \in S$ and $v_{j(r)} \in V-S$, for some r, $r = 1, \ldots, p$, then an optimal solution to problem (11) is obtained by setting

$$z^q = \begin{cases} 1 & \text{if } q = r, \\ 0 & \text{otherwise,} \end{cases} \tag{12}$$

$$y^q_{i,j} = \begin{cases} 1 & \text{if } q = r, v_i \in S,\ v_j \in V-S \quad \text{and} \quad (i,j) \in A', \\ 0 & \text{otherwise.} \end{cases}$$

(We say that (12) is the solution defined by the cut.) Therefore, the following result holds.

Theorem 5. *Every minimum D-cut defines an optimal solution to problem (11).*

We are now ready to prove that the polynomial formulation (9) is indeed a valid formulation for the minimum D-cut problem.

Theorem 6. *There exist a binary optimal solution y to the linear program (9), and $r \in \{1, \ldots, p\}$, such that*

$$y^q_{i,j} = 0, \quad \forall q = 1, \ldots, p,\ q \neq r, \text{ and } (i,j) \in A', \text{ and}$$

the set of arcs (v_i, v_j) satisfying $y^r_{i,j} = 1$, defines a $v_{i(r)} - v_{j(r)}$ cut of minimum capacity. This cut is a minimum D-cut.

Proof: We apply the same idea used in the proof of Theorem 1. Let y be an arbitrary feasible solution to problem (9). For each q, $q = 1, \ldots, p$, define $z^q = y^q_{i(q),j(q)}$. Using the metric constraints in (9) repeatedly, we can show (as in the proof of Theorem 1), that the pair y, z constitutes a feasible solution to problem (11). Thus, we conclude that problem (9) is a restriction of problem (11). Next, consider an optimal solution to problem (11), which is defined by a D-cut of minimum capacity (as in (12)). Suppose that the latter is also a $v_{i(r)} - v_{j(r)}$ cut of minimum capacity for some r, $r = 1, \ldots, p$. Using Theorem 1 we may conclude without loss of generality that the y vector representing this cut is feasible for problem (9) as well. Since problem (9) is a restriction of problem (11), the proof is now complete. □

Concluding Remarks.

We considered above several linear systems which formulate the mimimum D-cut problem on directed graphs. In all the formulations each arc of the graph is represented by $|D|$ real variables. At optimality all these $|D|$ variables, but one, are zero, and the non-zero variable is equal to 1. We note two instances where the number of arc variables can be reduced. The first case is that of an *s-cut* defined by some distinguished node, say s, and $D = \{[s,t]|t \in V,\ t \neq s\}$.

An exponential formulation of this problem follows from the work of Edmonds (1970,1973). A spanning arborescence in G rooted at s (an *s-arborescence*) is a spanning tree of the underlying undirected graph of G, having the properties:

(i) each node of G other than s has just one arc of the arborescence directed toward it,

(ii) no arc of the arborescence is directed toward s.

To simplify the notation suppose, as above, that $s = v_1$. If R is an s-arborescence , we also view R as a subset of A, and let $R' = \{(i,j) : (v_i, v_j) \in R\}$.

Consider the following linear program.

$$\begin{aligned} \text{Minimize} \quad & \sum_{(i,j)\in A'} c_{i,j} y_{i,j} \\ \text{s.t.} \quad & \sum_{(i,j)\in R'} y_{i,j} \geq 1, \quad \forall v_1 - \text{arborescence } R\ , \\ & y \geq 0\ . \end{aligned} \tag{13}$$

It follows from Edmonds (1970,1973) that problem (13) has an integer optimal solution vector, which is a minimum v_1-directed cut. Moreover, when the arc capacities $\{c_{i,j}\}$ are integer, the dual of (13) also has an integer optimal solution. Note that the dual corresponds to the following packing problem of v_1-arborescences: Assign each v_1-arborescence a weight in such a way that the sum of all the weights of v_1-arborescences that contain the arc (v_i, v_j) does not exceed $c_{i,j}$. The maximum packing of v_1-arborescences is a packing in which the sum of the weights assigned to v_1-arborescences is as large as possible.

The second case is that of finding the minimum unrestricted cut of an undirected graph. (The *unrestricted cut problem* is defined by $D = \{[v_i, v_j]|i,j = 1,\ldots,n,\ i \neq j\}$.) When the graph is undirected, the unrestricted cut problem can be converted into an s-

directed cut problem by replacing each undirected edge by two oppositely directed arcs, and assigning each one of them the capacity of the original edge. The terminal node s can be selected arbitrarily. Thus, we obtain a linear system for the minimum unrestricted cut on an undirected graph, where each edge of the graph is represented by two variables.

The reader is referred to Ahuja, Magnanti and Orlin (1993) for a survey of the most efficient algorithms for finding a minimum $s-t$ cut. Generally, a minimum D-cut of a graph can be found by solving $O(\min(n,|D|))$ $s-t$ cut problems. However, direct algorithms for solving the minimum unrestricted cut problem have recently been proposed. Mansour and Schieber (1989) considered the case of unit arc capacities, and gave an $O(nm)$ algorithm for this case. (The same bound for undirected graphs was proposed in Karzanov and Timofeev (1986) and in Matula (1987).) Gabow (1991) used a matroid approach and designed an algorithm for this case with an improved bound of $O((m^2/n)\log(n^2/m))$. Hao and Orlin (1992), and Nagamochi and Ibaraki (1992) considered graphs with general capacities and presented $O(nm\log(n^2/m))$ and $O(nm+n^2\log n)$ algorithms for the directed and undirected cases, respectively. Finally, Karger and Stein (1993) gave an $O(n^2\log^3 n)$ randomized algorithm for undirected graphs with general capacities.

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