Nested Matrices and the Existence of Least Majorized Elements

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ABSTRACT

We characterize the matrices A for which $X(b)=\{x|x\in R^n,\ x\geqslant 0,\ Ax\geqslant b,\ \Sigma_{i=1}^n x_i=1\}$ contains a least majorized element for all vectors b satisfying $X(b)\neq\varnothing$.

Given a vector $x=(x_1,\dots,x_n)'$, let $\tilde{x}$ denote the n-dimensional vector obtained by arranging the coordinates of x in decreasing order. A vector y is said to be majorized by a vector x if for $i=1,\dots,n$, $\Sigma_{j=1}^i \tilde{y}_j \leqslant \Sigma_{j=1}^i \tilde{x}_j$, with equality holding for $i=n$. Hardy, Littlewood, and Polya [2] proved that y is majorized by x if and only if $y=Sx$ for some doubly stochastic matrix S. An extended real-valued function f on R^n, $f>-\infty$, is called Schur convex if $x,y\in R^n$ and x majorized by y implies $f(x)\leqslant f(y)$. For example, a symmetric, quasiconvex function is Schur convex. Examples of decision models based on Schur convex functions can be found in [1; 3, 4, 5, 6].

It follows that if x is a least majorized element of a set $X\subseteq R^n$, i.e. $x\in X$ and x is majorized by all $y\in X$, then the minimum of any Schur convex function defined on X is attained at x. This fact motivates the study of sets containing a least majorized element. In this paper we focus on polyhedral sets of the form

$$X(b)=\left\{x|x\in R^n, x\geqslant 0, Ax\geqslant b,\ \sum_{i=1}^n x_i=1\right\}$$

and characterize the matrices A for which $X(b)$ contains a least element for all vectors b for which $X(b)$ is nonempty. The only results, that we know of, regarding the existence of least elements are in the work of Veinott [6], who studied an extension of the majorization concept. In fact, the necessity part of our characterization will follow from Veinott's theorem.

To present our main result we need the following definitions.

Let $A=(a_{ij})$ be an $m\times n$ 0,1 matrix. For $i=1,\dots,m$ let S_i be the support of the ith row, i.e. $S_i=\{j \mid a_{ij}=1\}$. Also denote $N=\{1,2,\dots,n\}$, and $M=\{1,2,\dots,m\}$.

DEFINITION.

(1) A 0,1 matrix A is called subnested if for any $i,k\in M$ one of the following holds:

(i) $S_i\cap S_k=\varnothing$,
(ii) $S_i\cup S_k=N$,
(iii) $S_i\subseteq S_k$,
(iv) $S_k\subseteq S_i$.

It is nested if for any $i,k\in M$ one of (i), (iii), (iv) holds.

(2) A real $m\times n$ matrix B is quasinested if there exist a subnested matrix A and diagonal matrices D^1 and D^2, $D^1\geqslant 0$, such that $B=D^1A+D^2E$, where E is the $m\times n$ matrix with all entries 1.

The main result is formulated as follows:

THEOREM 1. *A is quasinested if and only if $X(b)$ contains a least majorized element for all b for which $X(b)$ is nonempty.*

To prove the theorem we need the following lemma.

LEMMA 1. *Given numbers $a_1,\dots,a_n$, suppose that for any real d the nonemptiness of*

$$X(d)=\left\{x \mid x\in R^n,\ \sum_{i=1}^{n} a_i x_i\geqslant d,\ \sum_{i=1}^{n} x_i=1,\ x\geqslant 0\right\}$$

implies the existence of a least majorized element in $X(d)$. Then the sequence $\{a_1,\dots,a_n\}$ contains at most two distinct elements.

Proof. By permuting the variables and using the constraint that the variables sum to 1, we may assume without loss of generality that

$$a_1\geqslant a_2\geqslant \cdots \geqslant a_n>0.$$

Further, assume by contradiction that for some $1 \leq k < l < m \leq n$, $a_1 = \cdots = a_k > a_{k+1} = \cdots = a_l > a_{l+1} = \cdots = a_m$. It is then easily observed that if y belongs to $X(d)$, so does $\tilde{y}$. Thus, if x is a least majorized element, we assume that $x = \tilde{x}$. Moreover, without loss of generality we can assume that $x_1 = \cdots = x_k \geq x_{k+1} = \cdots = x_l \geq x_{l+1} = \cdots = x_m$. It also follows from the minimality of x that $(x_1, x_{k+1}, x_{l+1}, x_{m+1}, x_{m+2}, \dots, x_n)$ simultaneously minimizes the two linear functions ky_1 and $ky_1 + (l-k)y_{k+1}$ over the set of all $(y_1, y_{k+1}, y_{l+1}, y_{m+1}, y_{m+2}, \dots, y_n) \geq 0$ satisfying

$$ka_1 y_1 + (l-k)a_{k+1}y_{k+1} + (m-l)a_{l+1}y_{l+1} + \sum_{i>m} a_i y_i \geq d, \tag{1}$$

$$ky_1 + (l-k)y_{k+1} + (m-l)y_{l+1} + \sum_{i>m} y_i = 1 \tag{2}$$

$$y_1 \geq y_{k+1} \geq y_{l+1} \geq y_{m+1} \geq y_{m+2} \geq \cdots \geq y_n. \tag{3}$$

We now choose d such that

$$\frac{a_1 k + a_{k+1}(l-k)}{l} > d > \frac{a_1 k + a_{k+1}(l-k) + a_{l+1}(m-l)}{m} \tag{4}$$

Due to our assumptions on the sequence $\{a_1, a_2, \dots, a_n\}$, such a scalar d exists and $X(d)$ is nonempty. We will contradict the existence of the least majorized vector x by showing that every vector minimizing ky_1 subject to (1)–(3) satisfies $y_1 = y_{k+1}$, while every vector minimizing $ky_1 + (l-k)y_{k+1}$ satisfies $y_1 > y_{k+1}$. Consider first the function ky_1, and let $(\bar{y}_1, \bar{y}_{k+1}, \bar{y}_{l+1}, \bar{y}_{m+1}, \bar{y}_{m+2}, \dots, \bar{y}_n)$ be a minimum point. Suppose that $\bar{y}_1 > \bar{y}_{k+1} \geq \bar{y}_{l+1}$. If $\bar{y}_{l+1} = 0$, then $\bar{y}_i = 0$ $\forall i \geq l+1$, and $k\bar{y}_1 + (l-k)\bar{y}_{k+1} = 1$. Hence $\bar{y}_1 > 1/l$. On the other hand, the solution vector defined by $y_1 = y_{k+1} = 1/l$, $y_i = 0$ $\forall i \geq k+1$ is in $X(d)$, from (4), and therefore contradicts the minimality of $\bar{y}$. Thus, suppose

$$\bar{y}_1 > \bar{y}_{k+1} \geq \bar{y}_{l+1} > 0.$$

Let $\bar{y}_i$ be the smallest positive coordinate of $\bar{y}$. Subtract $\varepsilon > 0$, sufficiently small, from $\bar{y}_{l+1}, \bar{y}_{m+1}, \dots, \bar{y}_i$ and increase $\bar{y}_{k+1}$ so that (2) and (3) are still satisfied, with the first component, $\bar{y}_1$, strictly greater than the second component of the perturbed solution.

The monotonicity of the sequence $\{a_1, a_2, \dots, a_n\}$ implies that this perturbed solution, still having $\bar{y}_1$ as its largest component, will satisfy (1) as a strict inequality. Thus we can reduce $\bar{y}_1$ and increase the (perturbed) second coordinate so that (1)–(3) are met, hence contradicting the minimality of $\bar{y}_1$. We therefore have to have $\bar{y}_1=\bar{y}_{k+1}$.

Next we turn to a solution minimizing $ky_1+(l-k)y_{k+1}$, subject to (1)–(3). Again we use $\bar{y}$ to denote such a solution, and suppose that $\bar{y}_1=\bar{y}_{k+1}\geqslant\bar{y}_{l+1}$. First we show that if $\bar{y}_1=\bar{y}_{k+1}=\bar{y}_{l+1}\bar{y}$ does not satisfy (1). Note that $\bar{y}_1=\bar{y}_{k+1}=\bar{y}_{l+1}\leqslant 1/m$. The left-hand side of (1) will not decrease if we decrease all but the first three components of $\bar{y}$ to zero and increase $\bar{y}_1, \bar{y}_{k+1}, \bar{y}_{l+1}$ to $1/m$. But then, the value of the left-hand side becomes $\{a_1k+a_{k+1}(l-k)+a_{l+1}(m-l)\}/m$, which is smaller than d by (4). Thus the original solution was not feasible. Therefore, we assume $\bar{y}_1=\bar{y}_{k+1}>\bar{y}_{l+1}$. We now perturb $\bar{y}$ to reduce $k\bar{y}_1+(l-k)\bar{y}_{k+1}$. Increase $k\bar{y}_1$ by ε_1, and reduce $(l-k)\bar{y}_{k+1}$ by $\varepsilon_2>\varepsilon_1$. To maintain (2) increase $(m-l)\bar{y}_{l+1}$ by $\varepsilon_2-\varepsilon_1$. The perturbed solution meets (1) if $a_1\varepsilon_1-a_{k+1}\varepsilon_2+a_{l+1}(\varepsilon_2-\varepsilon_1)\geqslant 0$ or $\varepsilon_1\geqslant \varepsilon_2(a_{k+1}-a_{l+1})/(a_1-a_{l+1})=\bar{\varepsilon}_2$. $\bar{\varepsilon}_2<\varepsilon_2$, since $a_{k+1}<a_1$. If we choose $\varepsilon_2>0$ such that (3) is satisfied and then choose ε_1 such that $\bar{\varepsilon}_2<\varepsilon_1<\varepsilon_2$, the perturbed solution is feasible, but the objective $ky_1+(l-k)y_2$ is reduced, contradicting the optimality of $\bar{y}$. Thus we have to have $\bar{y}_1>\bar{y}_{k+1}$, and the existence of a least majorized vector is contradicted. ■

COROLLARY 1. *Let $A=(a_{ij})$ be an $m\times n$ real matrix such that $X(b)$ contains a least element for any b for which $X(b)$ is nonempty. Given $1\leqslant i\leqslant m$, there exist e_i and f_i such that $a_{ij}\in\{e_i, f_i\}$ for all $1\leqslant j\leqslant n$.*

Proof. Given i, $1\leqslant i\leqslant m$, we can choose a vector b with b_k, $k\neq i$, sufficiently small that all but the ith constraints of the system $Ax\geqslant b$ become redundant. The result will then follow from the previous lemma. ■

LEMMA 2. *Let $A=(a_{ij})$ be a $2\times n$ 0,1 matrix such that $X(b)$ contains a least element for all b for which $X(b)$ is nonempty. Then A is subnested.*

Proof. Suppose that A is not subnested. Then by permuting the variables we may assume the existence of $1\leqslant k$, $1\leqslant l$, $1\leqslant m$ and $k+l+m<n$, with

$$a_{ij}=1 \qquad \text{for}\quad i=1,2,\quad j=1,\dots,k,$$

$$a_{1j}=1 \qquad \text{for}\quad j=k+1,\dots,k+l,$$

$$a_{2j}=1 \qquad \text{for}\quad j=k+l+1,\dots,k+l+m,$$

$$a_{1j}=0 \qquad \text{for}\quad j>k+l,$$

$$a_{2j}=0 \qquad \text{for}\quad k<j\leqslant k+l, \qquad n\geqslant j>k+l+m.$$

If x is a least majorized element of $X(b)$, we may assume with no loss of generality that

$$x_j=\begin{cases} u_1, & 1\leqslant j\leqslant k,\\ u_2, & k<j\leqslant k+l,\\ u_3, & k+l<j\leqslant k+l+m,\\ u_4, & k+l+m<j\leqslant n,\end{cases}$$

and $u_1\geqslant u_2\geqslant u_4$, $u_1\geqslant u_3\geqslant u_4$. Furthermore (u_1,u_2,u_3,u_4) simultaneously minimizes the functions ky_1 and $ky_1+ly_2+my_3$ subject to the constraints

$$ky_1+ly_2 \geqslant b_1, \tag{5}$$

$$ky_1 + my_3 \geqslant b_2, \tag{6}$$

$$ky_1+ly_2+my_3+(n-k-l-m)y_4=1, \tag{7}$$

$$y_1\geqslant y_2,\qquad y_1\geqslant y_3,\qquad y_2\geqslant y_4,\qquad y_3\geqslant y_4\geqslant 0. \tag{8}$$

Suppose that we choose $b_1=b_2=(2k+1)/(2k+2)$. Adding (5) and (6) and using (7), we obtain

$$2b_1\leqslant 2ky_1+ly_2+my_3\leqslant 1+ky_1, \tag{9}$$

or

$$y_1\geqslant\frac{2b_1-1}{k}=\frac{1}{k+1}.$$

Also,

$$y_2+y_3+y_4\leqslant ly_2+my_3+(n-k-l-m)y_4$$

$$=1-ky_1\leqslant\frac{1}{k+1}\leqslant y_1,$$

and we can replace (8) by

$$y_2\geqslant y_4,\qquad y_3\geqslant y_4\geqslant 0. \tag{8'}$$

Turning first to the minimizer of ky_1, we note that $y_1 \geqslant y_2 + y_3 + y_4$ implies that (u_1, u_2, u_3, u_4) minimizes ky_1 subject to the constraints

$$\begin{aligned} ky_1 + ly_2 \quad &\geqslant b_1 \\ ky_1 \quad + my_3 &\geqslant b_1 \\ 1 \geqslant ky_1 + ly_2 + my_3&, \\ y_i \geqslant 0, \quad i = 1,2,3&. \end{aligned}$$

The minimum is attained at $ky_1 = 2b_1 - 1$, i.e. $u_1 = (2b_1 - 1)/k = 1/(k+1)$. From (9) it follows that $ku_1 + lu_2 = ku_1 + mu_3 = b_1$, and therefore

$$(ku_1, lu_2, mu_3, (n-k-l-m)u_4) = (2b_1 - 1, 1-b_1, 1-b_1, 0) \tag{10}$$

To contradict the existence of the minimal element x defined by (u_1, u_2, u_3, u_4), we show that x does not minimize the function $ky_1 + ly_2 + my_3$. The value of this function at x is 1.

Without any loss we assume that $l \geqslant m$, and consider the point (y_1, y_2, y_3, y_4) defined by

$$(ky_1, ly_2, my_3, ry_4) = (l+r)^{-1}(2b_1 l + b_1 r - l, l(1-b_1), l(1-b_1), r(1-b_1)),$$

where $r = n-k-l-m$. It is easily verified that this point satisfies (5)–(8) and $ky_1 + ly_2 + my_3 = (l + b_1 r)/(l+r) < 1$. By that we have contradicted the existence of the least majorized element x. ■

COROLLARY 2. *Let $A = (a_{ij})$ be an $m \times n$ real matrix such that $X(b)$ contains a least element for any b with $X(b) \neq \varnothing$. Then A is quasinested.*

Proof. Using Corollary 1 and observing that points in $X(b)$ satisfy $\sum_{i=1}^{n} x_i = 1$, we may assume that no row of A is proportional to e, the vector of $1-s$. Furthermore, there exist diagonal matrices $D^1 > 0$, D^2 and a 0, 1 matrix A^1 with $A = D^1 A^1 + D^2 E$. Therefore it suffices to prove the subnestedness of A^1. The assumptions on $X(b)$ and the nonsingularity of D^1 imply that $\{x \mid x \geqslant 0,\ A^1 x \geqslant b^1,\ \sum_{i=1}^{n} x_i = 1\}$ contains a least element for any b^1 for which the set is nonempty. The subnestedness of A^1 then follows from Lemma 2, while noting that any constraint can be omitted if the respective right-hand side coefficient is chosen to be sufficiently small. ■

To complete our characterization of quasinested matrices, Theorem 1, we apply the results of Veinott [6]. Again, by using the transformation with the diagonal matrices (as in the above proof), we may assume that A itself is subnested. We show how to express the set $X(b)$ as

$$X(b)=\left\{x \mid x\geqslant 0,\, b^1\geqslant \tilde{A}x\geqslant b^2,\quad \sum_{i=1}^{n} x_i=1\right\}, \tag{11}$$

where $\tilde{A}$ is nested and $b^1\geqslant b^2\geqslant 0$. Considering A, we can clearly assume that for no i $S_i=N$ or $S_i=\varnothing$, and $S_i\neq S_k$ if $i\neq k$. If there is no pair $i,k\in M$ with $S_i\cup S_k=N$ and $S_i\cap S_k\neq\varnothing$, then let $\tilde{A}=A$; otherwise define

$$T(A)=\{(i,k) \mid 1\leqslant i<k\leqslant m,\, S_i\cap S_k\neq\varnothing,\, S_i\cup S_k=N\}.$$

Let $(i,k)\in T$, and let S_r be a maximal support containing S_i. Then either $(r,k)\in T$ or $(k,r)\in T$. It is easily observed that if we replace S_r by $S_r'=N-S_r$, i.e. replace the rth row $(a_{r1},\dots,a_{rn})$ by $(1-a_{r1},\dots,1-a_{rn})$, then the new matrix $\tilde{A}$ will still be subnested.

We claim that $|T(\tilde{A})|\leqslant|T(A)|-1$. Suppose that for some t $S_r'\cap S_t\neq\varnothing$ and $S_t\cup S_r'=N$. Then $S_r\subseteq S_t$. But $S_r\neq S_t$, thus contradicting the maximality of S_r. This shows that $T(\tilde{A})\subseteq T(A)$. But the pair (k,r) [or (r,k)] is not in $T(\tilde{A})$. Continuing this process will yield a nested matrix. We note that each time we replace a support by its complement, the corresponding constraint, say $\sum_{j=1}^{n}a_{rj}x_j\geqslant b_j$, is replaced by $\sum_{j=1}^{n}(1-a_{rj})x_j\leqslant 1-b_j$.

Let $X(b)$ be given by (11) with $\tilde{A}$ nested. Furthermore we may assume that the set of supports of $\tilde{A}$, $\{S_1,\dots,S_m\}$, satisfies

$$S_i\neq\varnothing,\quad S_i\neq N \qquad \text{for}\quad 1\leqslant i\leqslant m,$$

$$S_i\neq S_k \qquad \text{for}\quad 1\leqslant i<k\leqslant m.$$

[Under these assumptions $m\leqslant 2(n-1)$.] Since we can add the constraints $0\leqslant x_i\leqslant 1$, $1\leqslant i\leqslant n$, we also assume that the supports corresponding to these constraints are present in $\{S_1,S_2,\dots,S_m\}$.

To apply the results of [6], we define a directed network as follows. For convenience denote $N=S_{m+1}$. Associate a node v_i with S_i, $1\leqslant i\leqslant m+1$. Connect v_i and v_k with an arc going from v_i to v_k if and only if $S_i\subseteq S_k$ and there is no S_l, $l\neq i,k$, with $S_i\subseteq S_l\subseteq S_k$. The generated network is a directed tree with n tips corresponding to the n supports containing one element. Also note that with the exception of v_{m+1} there is exactly one outgoing arc from each node. We now augment the directed tree with one more node, v_0, which is then connected to each one of the n tips of the tree. Finally we represent the constraints of (11) as a special case of the flow problem in [6]. The representation is similar to that given in Sec. 4 of [6]. We define the exogenous flows at the nodes and the arc capacities inductively, starting with the n tip nodes and the arcs leaving these nodes. The lower bounds on the flows are zeroes. [Recall that node v_i, $1 \leq i \leq m$, corresponds to the ith row of $\tilde{A}$ in (11).] For each tip node v_i define the exogenous outflow $c_i = b_i^2$. The upper bound on the flow on the arc leaving v_i will be given by $b_i^1 - b_i^2$. Now let v_i be a nontip node, and let T_i be the set of nodes v_k, $k \neq i$, on all the directed paths from the tips to v_i. Define the exogenous outflow at v_i by $c_i = \max(0, b_i^2 - \Sigma_{v_k \in T_i} c_k)$. The upper bound on the flow on the arc outgoing from v_i will be given by $b_i^1 - c_i - \Sigma_{v_k \in T_i} c_k$. Finally turning to v_{m+1}, we note that if $\Sigma_{v_k \in T_{m+1}} c_k > 1$, then $X(b)$ is empty. Otherwise define $c_{m+1} = 1 - \Sigma_{v_k \in T_{m+1}} c_k$.

Note that the variables x_i in (11) correspond to the flows from v_0 to the n tip nodes of the network. Furthermore, using the definition of the exogenous out flows, c_i, a simple inductive argument on the nodes of the network shows that each feasible solution to (11) induces a feasible flow in the network and vice versa.

Having formulated the constraints of (11) as a flow problem, Theorem 1 of [6] ensures the existence of a least majorized element in $X(b)$, provided $X(b)$ is not empty. This completes the proof of our characterization of quasinested matrices.

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