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OPTIMIZATION PROBLEMS WITH ALGEBRAIC SOLUTIONS: QUADRATIC FRACTIONAL PROGRAMS AND RATIO GAMES

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A mathematical program with a rational objective function may have irrational algebraic solutions even when the data are integral. We suggest that for such problems the optimal solution will be represented as follows: If λ^* denotes the optimal value there will be given an interval I and a polynomial $P(\lambda)$ such that I contains λ^* and λ^* is the unique root of $P(\lambda)$ in I. It is shown that with this representation the solutions to convex quadratic fractional programs and ratio games can be obtained in polynomial time.

Key words: Algebraic Numbers, Algebraic Optimization Problems, Quadratic Fractional Programming, Ratio Games.

1. Introduction

It is well known that the optimal value of a mathematical program with a rational objective function may be irrational even when the data defining the problem are integral. For example, minimizing the ratio between a convex quadratic and a linear function over some polyhedron yields an optimal value which is a root of some quadratic [3]. Thus, a finite procedure to find the 'exact' optimal objective value, allowing only the four operations +, −, * and /, does not exist. How does one measure the efficiency of algorithms designed to solve such optimization problems? Are there finite or even efficient (i.e., polynomial) algorithms which use only the above four operations on rational data, and produce a 'reasonable' representation of the irrational solution? These are only some of the relevant issues. We respond to these questions in the case when the optimal value of the objective is algebraic. In this case a possible representation of the solution may be a polynomial with integral coefficients having this solution as one of its roots. (One may even sharpen this representation and ask for such a polynomial with a minimal degree.) Furthermore, to achieve uniqueness it is desirable that the polynomial is attached with an interval such that the sought for value is the unique (distinct) root of the polynomial in the interval.

In this work we focus on two classes of problems that will be solved efficiently with respect to the above representation of an optimal solution. To explain our solution approach we will use the following general scheme.

Consider an interval $[s, t]$, s, t integer, and a sequence of distinct points $\{\lambda_1, \dots, \lambda_k\}$ where $s \leq \lambda_i \leq t$, $i = 1, \dots, k$. Suppose that the points are not explicitly given (e.g. roots of some equation), and let Δ be a rational lower bound on the distance between two elements of the sequence. For any real s^*, unspecified, in $[s, t]$, a subinterval of length $< \Delta$, containing s^* (and therefore at most one element of the sequence), can be obtained in $O(\log((t-s)/\Delta))$ queries of the form: "is $y \leq s^*$?". In particular, when s^* is itself some member of the sequence, which is given implicitly by some property, the above process enables us to isolate s^* from the rest of the elements. Of course, the crux of the process is an explicit value for Δ, and an efficient method to answer the query.

In this paper we study quadratic fractional programming problems and ratio games whose solution amounts to a search of the above nature. The sequence will correspond to the set of real roots of all nonzero integral polynomials of a bounded degree and a bounded coefficient size, and s^* will be the respective optimal value.

We will use results on algebraic numbers [1, 2, 8, 20] to obtain lower bounds on the distance between distinct elements of the above set. We will then apply the above approach to solve the following problems. The quadratic fractional program:

$$\begin{aligned} &\text{Minimize} \quad \frac{\frac{1}{2}x'Cx + b'x + a}{\frac{1}{2}x'Dx + c'x + d} \\ &\text{s.t.} \quad x \in S = \{x \in \mathbb{R}^n \mid Ax \geq p,\ x \geq 0\}, \end{aligned} \tag{1}$$

where S is nonempty and the denominator of the objective is positive for all x in S. We assume that the quadratics in the numerator and the denominator are convex and concave respectively. Also, the numerator is nonnegative for all x in S. The dimension of A is $m \times n$. The ratio game problem:

Given $m \times n$ matrices $A = (a_{ij})$, $B = (b_{ij})$ ($b_{ij} > 0$), find minimax solutions (the optimal value and optimal strategies) for a game with payoff function $P(x, y) = x'Ay/x'By$, i.e.,

$$\begin{aligned} &v = \operatorname*{Min}_{y} \operatorname*{Max}_{x} P(x, y) \\ &\text{s.t.} \quad \sum_{i=1}^{m} x_i = 1, \qquad \sum_{j=1}^{n} y_j = 1, \\ &\qquad x \geq 0,\ y \geq 0. \end{aligned} \tag{1a}$$

There are numerous papers in the literature dealing with general fractional programming and problem (1) in particular. Finite (nonpolynomial) algorithms for (1) are presented in [12, 28]. Superlinearly convergent infinite algorithms are described in [6, 10, 11, 27, 29, 31]. For a survey on algorithms in fractional programming see [30]. For previous works on problem (1a) see [5, 32].

Let λ^* denote the optimal value of the objective in (1), or the value v of the game (1a). We will first show that λ^* is indeed algebraic and then present an algorithm that finds the following in polynomial time.

(i) An interval I containing λ^*.

(ii) An integral polynomial $P(\lambda)$ such that λ^* is the unique root of $P(\lambda)$ in I, and the restriction of $P(\lambda)$ to I is either strictly monotone, or strictly unimodal with respect to λ^*.

The coefficients of the polynomial $P(\lambda)$ as well as the endpoints of I will have a binary encoding which is polynomial in the length of the input of the respective problem.

The outline of the paper is as follows. Section 2 presents results on algebraic numbers, and in particular lower bounds on the distance between distinct such numbers. Section 3 is devoted to the convex quadratic fractional problem. A detailed discussion on the polynomial procedure solving this problem is presented there. Section 4 demonstrates that the ratio game problem possesses the properties required for the application of the scheme presented in the previous section. Section 5 discusses further applications of the results on algebraic numbers to optimization problems. In particular, the solution to a convex piecewise quadratic function (expressed as the maximum of n convex quadratics) will be given a 'quadratic representation' in $O(n)$ time. The final section comments on computational aspects related to iterative procedures that locate roots of polynomials. We present there a polynomial algorithm to compute the multiplicity of a real root of a polynomial.

We would like to emphasize that our polynomial methods are mainly of theoretical interest at this time, since they are based on the use of ellipsoidal algorithms [14, 15], which are currently considered to be impractical for real applications. For example, to solve the quadratic fractional program our procedure requires the solution of $O(n^5L)$ quadratic programs, where n is the number of variables and L is the length of a binary encoding of the input. The polynomial complexity is assured if each quadratic program is solved by an ellipsoid algorithm [15]. (Of course, for practical applications one may choose to use standard simplex type algorithms to solve the quadratic programs.)

Generally, it should be pointed out that the use of ellipsoidal methods is only for purposes of proving complexity results, and to act as a catalyst to discover new and more efficient algorithms.

2. Results on algebraic numbers

A complex number α is called algebraic if there exists a nonzero polynomial $P(x)$ with integer coefficients such that $P(\alpha)=0$. A polynomial with integer coefficients is called primitive if the greatest common divisor (g.c.d.) of its coefficients is 1. $g_\alpha(x)$ will denote the unique (module sign) nonzero primitive polynomial of minimum degree such that $g_\alpha(\alpha)=0$. The height of a polynomial is the maximum of the absolute values of its coefficients.

The degree of an algebraic number α, $\deg(\alpha)$, is the degree of $g_\alpha(x)$ and the height of α, $h(\alpha)$, is defined to be the height of $g_\alpha(x)$. In particular, when α is rational, then its degree is 1 and its height is bounded by the maximum of the absolute values of its numerator and denominator. (The height is equal to this bound if and only if the numerator and the denominator are relatively prime.) The following results are well known and can be found, for example, in [1, 2, 8, 17, 20].

Lemma 1. *Let $H(x)$ be a nonzero polynomial with integer coefficients, and let α be such that $H(\alpha)=0$. Let n and h denote the degree and height of α respectively. Then $|\alpha|\leq nh$, and $g_\alpha(x)$ divides $H(x)$ (over the polynomials with integer coefficients).*

Theorem 1 [20]. *Let α and β be distinct algebraic numbers. Then*

$$|\alpha-\beta|\geq(2(\deg(\alpha)+1)h(\alpha))^{-\deg(\beta)}(2(\deg(\beta)+1)h(\beta))^{-\deg(\alpha)}. \tag{2}$$

Lemma 2 [1]. *Let $P(x)$ be a polynomial with degree n and height h, and let $P=P_1P_2\cdots P_k$ where $P_j(x)$ is a polynomial with height h_j. Then,*

$$e^{-n}h_1h_2\cdots h_k\leq h\leq n^{k-1}h_1h_2\cdots h_k. \tag{3}$$

We use the above to prove the following result.

Theorem 2. *Let $T(n,h)$ denote the finite set of roots of all nonzero polynomials with integer coefficients, whose degree is bounded by n and whose height is at most h. Let α_1 and α_2 be distinct elements in $T(n,h)$. Then,*

$$|\alpha_1-\alpha_2|\geq(2(n+1)\,\mathrm{e}^nh)^{-2n}. \tag{4}$$

Proof. Let $P_i(x)$ be a nonzero polynomial with integer coefficients such that its degree is at most n and its height is at most h. Suppose that $P_i(\alpha_i)=0$. Then, from Lemma 1 $g_{\alpha_i}(x)$ divides $P_i(x)$, and from Lemma 2 we obtain $h(\alpha_i)\leq \mathrm{e}^nh$. Clearly, $\deg(\alpha_i)\leq n$. Using these bounds on $h(\alpha_i)$ and $\deg(\alpha_i)$ (4) is directly obtained from (2). □

3. Quadratic fractional programming

Consider the quadratic fractional programming problems (1) defined in the Introduction. To simplify the presentation we add the assumption that the minimum of (1) is attained, e.g., S is bounded. (See the discussion at the end of the section regarding this assumption.) For a real λ define the (parametric) quadratic program

$$f(\lambda)=\underset{x\in S}{\text{Minimize}}\,\{\tfrac{1}{2}x'(C-\lambda D)x+(b-\lambda c)'x+(a-\lambda d)\}. \tag{5}$$

The assumptions made in the Introduction with respect to (1) ensure that $f(\lambda)$ is a well defined, strictly decreasing concave function of λ. Furthermore, the minimum objective value of (1) is equal to λ^* defined by $f(\lambda^*)=0$ [6, 12]. Thus, solving (1) amounts to searching λ^*. The nonnegativity of the numerator of the objective in (1) implies $f(0)\geqslant 0$. Therefore, $\lambda^*\geqslant 0$ and we will focus now on the range $\lambda\geqslant 0$. For $\lambda\geqslant 0$, $f(\lambda)$ is the minimum of a convex quadratic so that the Karush–Kuhn–Tucker conditions for optimality are necessary and sufficient. These conditions yield a (parametric) linear complementarity problem of the form:

Find $z\geqslant 0$, $w\geqslant 0$, satisfying $w=M(\lambda)z+q(\lambda)$, $w'z=0$, where

$$M(\lambda)=\begin{bmatrix} C-\lambda D & -A' \\ A & 0 \end{bmatrix} \quad \text{and} \quad q(\lambda)=\begin{bmatrix} b-\lambda c \\ -p \end{bmatrix}. \tag{6}$$

$M(\lambda)$ is a $k\times k$ matrix with $k=m+n$. Focusing on basic solutions of (6), it was first proved by Ritter [28], and strengthened by Ibaraki et al. [12], that a given basis of (6) appears only finitely many times as a feasible basis when λ is continuously increased. Using Cramer's rule we note that each component of a basic solution is a rational function (ratio of two polynomials of degree $\leqslant n$), of λ. Therefore, $f(\lambda)$, $\lambda\geqslant 0$, is a piecewise rational function. (The breakpoints of $f(\lambda)$ are the values of λ at which there is a change of a complementary basis when λ is continuously increased from 0 to ∞.) Ibaraki et al. [12] exploit these properties of $f(\lambda)$ to find λ^*. They locate the 'piece' (interval) of $f(\lambda)$ containing λ^* by tracing all the basis changes occurring when λ is continuously increased from $\lambda=0$. It was shown that even for the case $C=D=0$, $f(\lambda)$ (which in this case is piecewise linear), may have an exponential number of breakpoints (see [22]). Thus the finite process in [12] for finding the optimal basis is not polynomial in L, the input length of the problem. L is defined as follows:

$$\begin{aligned} L=&\sum_{i=1}^{n}\sum_{j=1}^{n}(\log(|c_{ij}|+1)+\log(|d_{ij}|+1))+\sum_{i=1}^{m}\sum_{j=1}^{n}(\log(|a_{ij}|+1)) \\ &+\sum_{i=1}^{n}(\log(|c_i|+1)+\log(|b_i|+1))+\log(|a_i|+1)+\log(|d|+1)) \\ &+\sum_{i=1}^{m}\log(|p_i|+1)+\log mn+1, \end{aligned} \tag{7}$$

where $C=(c_{ij})$, $D=(d_{ij})$, $A=(a_{ij})$, $b=(b_i)$, $c=(c_i)$, and $p=(p_i)$.

In a recent paper [3] Chandrasekaran presented a polynomial algorithm for the special case $D=0$. In this case $f(\lambda)$ is piecewise quadratic, and each one of its breakpoints is a rational whose height has a binary encoding of length $O(L)$. Chandrasekaran searches over this set of breakpoints (as in Papadimitriou [24] and Reiss [26]), using the polynomial algorithm of [15] which computes $f(\lambda)$. In polynomial time he locates two consecutive breakpoints of $f(\lambda)$ that bracket λ^*. $f(\lambda)$ is quadratic on the interval determined by these two breakpoints and λ^* is one of its roots. Therefore the entire process for finding λ^* is polynomial provided that a solution of a quadratic is viewed as a constant time operation. Since λ^*, unlike the breakpoints of $f(\lambda)$, may be irrational, some assumption of this nature must be made. Note that such an operation is applied exactly once in [3].

When $D \neq 0$, the breakpoints of $f(\lambda)$ may also be irrational (see Ibaraki et al. [12]), and therefore the approach in [3] is not applicable anymore. Instead, we will apply a binary search over a range known to contain λ^*.

In the sequel the height of a rational number will denote the maximum of the absolute values of its numerator and denominator in the presentation which is being used. The rational number will not necessarily be given as a ratio of two relatively prime integers, although such a presentation can always be generated polynomially.

Let $Z^i(\lambda)$ denote a basic solution (not necessarily a feasible one), of (6), i.e., $Z^i(\lambda)$ is defined by some set of k independent columns of the matrix $[I, -M(\lambda)]$. As noted above each of the k components of $Z^i(\lambda)$ can be expressed as a ratio of two polynomials of degree $\leq n$. Furthermore by a proof similar to that in [7], we observe that each coefficient of these polynomials has an absolute value $\leq 2^L$. Therefore, the heights of the polynomials are at most 2^L.

A breakpoint of the function $f(\lambda)$ is a point where some basic solution becomes degenerate. Thus, each breakpoint of $f(\lambda)$ is a root of an integral polynomial whose degree is between 1 and n and its height is at most 2^L. Define T_1 to be the finite set of all real roots of integral polynomials whose degree is between 1 and n and their height is at most 2^L. T_1 contains all the breakpoints of $f(\lambda)$.

Let $\bar{x}^i(\lambda)$ be the subvector of $Z^i(\lambda)$ corresponding to the primal variables in (1). $\bar{x}^i(\lambda)$ consists only of those components of $Z^i(\lambda)$ corresponding to some column of the first n columns of $M(\lambda)$. The dimension of $\bar{x}^i(\lambda)$ is $\leq n$. Denote by $x^i(\lambda)$ the n-dimensional vector obtained from $\bar{x}^i(\lambda)$ by augmenting zeroes for those primal variables that are not part of the k-dimensional basic solution $Z^i(\lambda)$. Define

$$f_i(\lambda) = \tfrac{1}{2}x^i(\lambda)(C - \lambda D)x^i(\lambda) + (b - \lambda c)x^i(\lambda) + (a - \lambda d). \tag{8}$$

On each interval that does not contain a breakpoint of $f(\lambda)$ in its interior, $f(\lambda)$ coincides with some function $f_i(\lambda)$. In particular, λ^*, the unique zero of $f(\lambda)$ is also a zero of some function $f_i(\lambda)$.

A function $f_i(\lambda)$ is the sum of $(n(n+1)/2 + n + 1) = (n(n+3)/2 + 1)$ rational functions. Each of these rational functions is a ratio of two polynomials of degree $\leq (2n+1)$. (This is easily observed by considering a typical term of the quadratic form $x^i(\lambda)(C - \lambda D)x^i(\lambda)$.) The heights of these polynomials are bounded above by $(2n+1)^2 2^{3L}$ (Lemma 2). Being a sum of $(n(n+3)/2 + 1)$ rational functions, $f_i(\lambda)$ can now be represented as a ratio of two integral polynomials of degree $\leq (2n+1)(n(n+3)/2 + 1)$. (Use the relation

$$P_1(\lambda)/P_2(\lambda) + P_3(\lambda)/P_4(\lambda) = (P_1(\lambda)P_4(\lambda) + P_2(\lambda)P_3(\lambda))/(P_2(\lambda)P_4(\lambda))$$

to replace (inductively) a sum of two rational functions by a single rational function.) Using Lemma 2 we also conclude that the heights of these two integral polynomials is bounded by H, where

$$H=(n(n+3)/2+1)((2n+1)(n(n+3)/2+1))^{n(n+3)/2} \times((2n+1)^2 2^{3L})^{n(n+3)/2+1}. \tag{9}$$

Let T_2 be the finite set of real roots of all integral polynomials whose degree is bounded between 1 and $(2n+1)(n(n+3)/2+1)$ and their height is at most H. From the above discussion it follows that λ^* is in T_2. Also, the set T_1 defined above is a subset of T_2. Therefore, T_2 contains all the breakpoints of $f(\lambda)$. It will be convenient for the later part of the analysis to enlarge T_2, and consider instead the set T, defined as follows. T is the set of real roots of all integral polynomials whose degree is between 1 and $(2n+1)(n(n+3)/2+1)$ and their height is at most H_1.

$$H_1=(2n+1)(n(n+3)/2+1)H. \tag{10}$$

In particular, T contains also the roots of the derivatives of the polynomials used to define T_2.

Suppose without loss of generality that $n \geqslant 5$.

Theorem 3. *Let λ_i, $i=1, 2$, be two elements in T. Then,*
(i) $|\lambda_i| < A_1 = 2^{n^2(8L+3)}$.
(ii) *If* $\lambda_1 \neq \lambda_2$ *then* $|\lambda_1-\lambda_2| > A_2 = 2^{-16n^5(2L+n)}$.

Proof. For $n \geqslant 5$, the upper bound on the degree of the polynomials defining T can be replaced by $2n^3$. Also, since $n(n+3)/2+3 \leqslant n^2$, H_1, the upper bound on the heights can be replaced by

$$(2n^3)^{n^2-1}((2n+1)^2 2^{3L})^{n^2}.$$

(i) From Lemma 1, $|\lambda_i| \leqslant (2n^3)(2n^3)^{n^2-1}((2n+1)^2 2^{3L})^{n^2}$. Recalling that $n+1 \leqslant 2^L$, we obtain

$$|\lambda_i| < (2n^3)^{n^2}((2(n+1))^2 2^{3L})^{n^2} < 2^{n^2(8L+3)}.$$

(ii) From Theorem 2,

$$\begin{aligned}|\lambda_1-\lambda_2| &> (2(2n^3+1)\,\mathrm{e}^{2n^3}(2n^3)^{n^2-1}((2(n+1))^2 2^{3L})^{n^2})^{-4n^3}\\ &> (2(2(n+1)^3)^{n^2}\,\mathrm{e}^{2n^3}2^{2n^2}(n+1)^{2n^2}2^{3n^2L})^{-4n^3}\\ &> (2^{3n^2+1}\,\mathrm{e}^{2n^3}2^{8n^2L})^{-4n^3} > 2^{-4n^3(8n^2L+3n^3+3n^2+1)}\end{aligned}$$

where the last inequality follows from the relation $\mathrm{e}^2 < 2^3$. For $n \geqslant 5$, $3n^2+1 < n^3$. Therefore,

$$|\lambda_1-\lambda_2| > 2^{-16n^5(2L+n)}. \qquad \square$$

The next task is to find an interval that contains λ^* but no other distinct element of T. Recall that $\lambda^* \in T$ and $\lambda^* \geqslant 0$. For any given $\lambda \geqslant 0$, the query "is $\lambda \geqslant \lambda^*$?" is equivalent to "is $f(\lambda) \leqslant 0$?". Thus, by solving the convex quadratic program (5) the query can be answered.

Consider the sequence

$$\mu_i = iA_2, \quad i = 0, 1, 2, \ldots, A_1/A_2.$$

From Theorem 3 it follows that for each i the interval $[\mu_i, \mu_{i+1}]$ contains at most one element of T. Therefore, by applying a binary search on the sequence $\{\mu_i\}$, we identify an index t such that $f(\mu_t) \geqslant 0$ and $f(\mu_{t+1}) < 0$. (Without loss of generality we can assume that $f(\mu_t) > 0$ since otherwise $\lambda^* = \mu_t$.) λ^* is the only element of T in the interval $[\mu, \mu_{t+1}]$. The search for this interval amounts to $O(n^5 L)$ evaluations of the function $f(\lambda)$. $f(\lambda)$ is computed only for values in the sequence $\{\mu_i\}$. Each μ_i is a rational whose height is bounded by A_1/A_2. Thus, each evaluation of $f(\lambda)$ is performed in polynomial time [15] and therefore the interval $[\mu_t, \mu_{t+1}]$ is found in time which is polynomial in L.

At this point we note that if $[\mu_t, \mu_{t+1}]$ contains a breakpoint of $f(\lambda)$ it must coincide with λ^*.

Consider the vector $x(\mu_t)$ that yields $f(\mu_t)$ by the algorithm in [15]. Using linear programming methods $x(\mu_t)$ can be augmented (in polynomial time) by values for the other variables to form a solution to the respective complementarity problem (6). Consider the subsystem of the linear system $w = M(\mu_t)z + q(\mu_t)$, $w \geqslant 0$, $z \geqslant 0$, obtained by deleting all the columns (variables) associated with zero components of this complementary solution. (Due to the condition $w'z = 0$, the subsystem will have at most k variables.) Clearly, each feasible solution to this linear subsystem is also a solution to the complementarity problem. In particular, a basic solution of the linear subsystem is a complementary basic solution. Since we already have a feasible solution to the linear subsystem, it is known [21] that a basic solution can be derived from it in time which is polynomial in the size of the matrix. Let $Z(\mu_t)$ denote such a complementary basic solution. We assume without loss of generality that the positive components of $x(\mu_t)$ are part of the basic variables in $Z(\mu_t)$, and they correspond to independent columns of $[I, -M(\mu_t)]$. Next, for each one of the positive components of $x(\mu_t)$ we obtain (again, in polynomial time), the respective rational function expressing this component as a function of λ. $\lambda = \mu_t$ is not a breakpoint of $f(\lambda)$, since λ^*, which is inside $[\mu_t, \mu_{t+1}]$, is the only member of T in the interval. Therefore, for all $\lambda \in [\mu_t, \lambda^*]$, $f(\lambda)$ is determined by the rational functions corresponding to the positive components of $x(\mu_t)$. Substituting these functions in (8) we obtain the functional expression, denoted by $f_t(\lambda)$, which coincides with $f(\lambda)$ over $[\mu_t, \lambda^*]$. Next, we express $f_t(\lambda)$ (in polynomial time), as a ratio of two integral polynomials of degree $\leqslant (2n+1)(n(n+3)/2+1)$, i.e., $f_t(\lambda) = P_t(\lambda)/Q_t(\lambda)$. From the definition of T, we know that the only zero of $P_t(\lambda)$ or its derivative in $[\mu_t, \mu_{t+1}]$ is λ^*. Therefore, the restriction of $P_t(\lambda)$ to this interval is either strictly monotone or else strictly unimodal with respect to λ^*.

To summarize, we have described a procedure that finds in polynomial time, an interval containing λ^*, and an integral polynomial $P_t(\lambda)$ of degree $\leqslant 2n^3$ and height

$< A_1 = 2^{n^2(8L+3)}$, such that λ^* is the only zero of $P_t(\lambda)$ in the interval. Moreover, $P_t(\lambda)$ is strictly monotone, or else strictly unimodal, on this interval with respect to λ^*.

In order to simplify the presentation we have assumed at the beginning of this section that the minimum in (1) is attained. Suppose now that this assumption is removed, and let λ' denote the infimum of the fractional program. (The assumptions made in the Introduction imply that $\lambda' \geqslant 0$.) Defining $f(\lambda)$, $\lambda \geqslant 0$, as above, we allow $f(\lambda)$ to take on the value $-\infty$. It is well known that $f(\lambda)$ is a monotone (proper) concave function, i.e., there exists $\bar{\lambda} \geqslant 0$, such that $f(\lambda)$ is concave and strictly decreasing in $[0, \bar{\lambda}]$ and $f(\lambda) = -\infty$ for $\lambda > \bar{\lambda}$. The infimum of the fractional program, λ', is given by $\lambda' = \text{Sup}\{\lambda | f(\lambda) \geqslant 0\}$. In order to characterize the infimum, λ', and determine whether it is achieved, we use the notation of this section. We first find the interval $[\mu_t, \mu_{t+1}]$ such that $f(\mu_t) > 0$ and $f(\mu_{t+1}) < 0$. If $f(\mu_{t+1})$ is finite the minimum in (1) is achieved, and $\lambda' = \lambda^*$, where λ^* is characterized above.

Suppose that $f(\mu_{t+1}) = -\infty$. First we determine whether $\lambda' = \mu_t$. (Recall that $\mu_t \leqslant \lambda' \leqslant \mu_{t+1}$ and $f(\lambda') \geqslant 0$.) λ' is an algebraic number in the set T defined above. Therefore its height is at most $2^{n^2(8L+3)}$, and its degree is at most $2n^3$. μ_t is rational with height bounded above by

$$A_1/A_2 = 2^{16n^5(2L+n)+n^2(8L+3)} < 2^{20n^5(2L+n)}.$$

If $\lambda' > \mu_t$ we can apply Theorem 2 to conclude that $\lambda' - \mu_t > \mu' = 2^{-88n^8(2L+n)}$. Thus, $\lambda' > \mu_t$ if and only if $f(\mu' + \mu_t) \geqslant 0$. Computing $f(\mu' + \mu_t)$ (by [15]) will determine whether $\lambda' > \mu_t$. Note that since $f(\mu_t) > 0$, the case $\lambda' = \mu_t$ implies that the minimum in (1) is not attained. Suppose now that $\lambda' > \mu_t$. In this case λ' is the only member of T in $[\mu_t, \mu_{t+1}]$. We can now construct and consider the rational function $f_t(\lambda) = P_t(\lambda)/Q_t(\lambda)$ as defined above. If the polynomial $P_t(\lambda)$ has a zero in $[\mu_t, \mu_{t+1}]$, this is $\lambda' = \lambda^*$, the minimum value of (1), as defined above. Otherwise, the minimum in (1) is not achieved. In this case the infimum, λ', can also be given a polynomial characterization. λ' is the unique root in $[\mu_t, \mu_{t+1}]$ of one of the (at most k) rational functions corresponding to the positive components in the basic solution $Z(\mu_t)$. (We omit the details since finding this rational function is very similar to the construction of $f_t(\lambda)$ above.)

As mentioned above, all these rational functions can be constructed in polynomial time. Furthermore, testing whether a polynomial has a zero in a given interval can also be done in polynomial time (see the last section). Thus, we conclude that in polynomial time we can construct a polynomial representation of the infimum of a fractional program, and determine whether the minimum in (1) is achieved.

4. Ratio games

Consider the ratio game problem defined in (1a). An infinite iterative solution technique converging to the solution v is presented in [32]. It is based on the following lemma.

Lemma 3. *A necessary and sufficient condition for v to be the value of $P(x, y)$ and $(\bar{x}, \bar{y})$ to be optimal strategies is that the two person, zero sum game with payoff matrix $A - vB$ has value zero and optimal strategies $(\bar{x}, \bar{y})$.*

The lemma suggests that the minimax solution can be computed by a method similar to that of Section 3. For each real λ define $f(\lambda)$ to be the value of the two person, zero sum game with payoff matrix $A - \lambda B$, i.e.,

$$\begin{aligned} f(\lambda) = \text{Max} \quad & u \\ \text{s.t.} \quad & x'(A - \lambda B) \geqslant u\bar{e}, \\ & \sum_{i=1}^{m} x_i = 1, \quad x \geqslant 0, \end{aligned} \tag{11}$$

where $\bar{e} = (1, \ldots, 1)$. Our task is to find v, the value of $P(x, y)$, which is (by Lemma 3) a root of $f(\lambda)$. To apply the technique presented in Section 3 we must show that the query "is $\lambda \geqslant v$?" can be answered efficiently (in terms of the input of the problem), and that v is an algebraic number.

Due to the positivity of the matrix B, $f(\lambda)$ is a continuous, monotone decreasing function. Furthermore, $f(\lambda_1) \geqslant 0$ and $f(\lambda_2) \leqslant 0$, where $\lambda_1 = \text{Min}_{i,j}(a_{ij}/b_{ij})$ and $\lambda_2 = \text{Max}_{i,j}(a_{ij}/b_{ij})$. $f(\lambda)$ is a solution to a linear program, and it can be computed polynomially [14] for a rational λ, whose height has a binary encoding which is polynomial in L_1, the input length of this problem,

$$L_1 = \sum_{i=1}^{m} \sum_{j=1}^{n} (\log(|a_{ij}| + 1) + \log(|b_{ij}| + 1)) + \log nm + 1.$$

Therefore, by the monotonicity of $f(\lambda)$, the query "is $\lambda \geqslant v$?" can be answered by computing $f(\lambda)$.

Turning to the algebraicness of v we appeal to [13, Theorem 2.4.3]. v must satisfy an equation of the form det $\bar{M} = 0$, where $\bar{M}$ is a submatrix of $A - vB$. Therefore v is an algebraic number whose degree is at most $\min(m, n)$, and its height has a binary encoding of length which is polynomial in L_1. [13, Theorem 2.4.3] which also represents extreme solutions to zero sum games, enables us to follow exactly the same arguments as in Section 3. Thus we conclude that $f(\lambda)$ is a continuous monotone piecewise fractional function of λ, and the technique used for solving the convex quadratic fractional program is also applicable here.

To sum up we have shown that the value of a ratio game can be represented as follows. In time which is polynomial in L_1, the input length of the problem, we will find:

(i) An interval I containing the value v.

(ii) A polynomial $P(\lambda)$, of degree $\leqslant \min(m, n)$, such that v is the unique root of $P(\lambda)$ in I, and the restriction of $P(\lambda)$ to I is either strictly monotone, or strictly unimodal with respect to v.

5. Further applications

In this section we discuss and comment on additional applications of the bounds on algebraic numbers presented in Section 2.

Consider the problem of minimizing a convex piecewise quadratic function. Formally, let

$$h(x) = \underset{i=1,\dots,n}{\text{Maximum}} \{h_i(x)\} \quad \text{where } h_i(x) = a_i x^2 + b_i x + c_i, \qquad a_i \geq 0. \qquad (12)$$

The minimum of $h(x)$, if it exists, is attained either at a point of intersection of two quadratics or at the minimum of one of the defining quadratics. It is shown in [19] that the minimum of $h(x)$ can be found in $O(n)$ time, assuming that the square root operation takes constant time. Specifically, the algorithm in [19] executes $O(n)$ square root operations and $O(n)$ comparisons between such roots.

Using the bounds between algebraic numbers, derived in Secion 2, we can show that it is not necessary to compute 'exact' roots. It suffices to approximate a root of a quadratic by some rational. (This rational can be obtained in polynomial time, and the length of its height is polynomial in the input length.) A comparison between roots of quadratics will be equivalent to a comparison between their respective approximating rationals. Thus, in $O(n)$ comparisons between approximating rationals we can find the two quadratics whose intersection point is a minimum of $h(x)$, or, depending on the case, the quadratic whose minimum is also the minimum of $h(x)$.

However, in cases like the above, where the algebraic numbers are of a low degree (e.g., roots of quadratics), and where the algorithm is based only on comparisons of such numbers, it is not always necessary to construct approximating rationals. We demonstrate this idea on the above convex quadratic minimization problem.

The algorithm in [19] is based only on comparisons between elements of the same set, of one of the following three sets.

(A_1) Roots of quadratics of the form $h_i(x) - h_j(x)$, $i, j = 1, \dots, n$, where $h_i(x) - h_j(x)$ is not identically zero.

(A_2) Algebraic numbers of the form $h_i(\alpha)$, where α is in A_1.

(A_3) Algebraic numbers of the form $h_i'(\alpha) = 2a_i\alpha + b_i$, where α is in A_1. (In fact, the numbers in A_3 are compared with 0 only.)

It can be shown [4] that each of the $O(n)$ comparisons performed by the algorithm in [19] amounts to a comparison between two roots of quadratics, whose integer coefficients are determined in constant time. The latter comparison can be executed in constant time. Suppose that $\alpha_i = p_i \pm \sqrt{q_i}$, $i = 1, 2$, where p_i and q_i are rational. To determine, for example whether "$\alpha_1 = p_1 + \sqrt{q_1} \geq p_2 + \sqrt{q_2} = \alpha_2$", we first test the sign of $q_1 - q_2$. Suppose, without loss of generality, that $q_1 \geq q_2$. If $p_2 - p_1 \leq 0$ then $\alpha_1 \geq \alpha_2$. If $p_2 - p_1 > 0$, then $\alpha_1 \geq \alpha_2$ if and only if $-2\sqrt{q_1 q_2} \geq (p_2 - p_1)^2 - (q_1 - q_2)$. Now, the validity of the latter is tested as follows. If $(p_2 - p_1)^2 - (q_1 - q_2) > 0$, the inequality is invalid, otherwise it holds if and only if $4q_1q_2 \leq ((p_2 - p_1)^2 - (q_1 - q_2))^2$. All other cases are similarly analyzed.

We conclude that one can drop the supposition in [19] that the square root operation takes constant time. Instead, in $O(n)$ comparisons and basic operations (i.e., $+$, $-$, and $*$) on rationals, we will identify a quadratic whose solution is a minimum point of $h(x)$, provided that the minimum exists. The existence of a finite minimum point of $h(x)$ is tested as follows. Each breakpoint of $h(x)$, α, is in A_1. From Lemma 1, $|\alpha| \leq 2D$, where

$$D = \operatorname*{Maximum}_{i,j=1,\dots,n} \text{Maximum}\,(|a_i - a_j|, |b_i - b_j|, |c_i - c_j|). \tag{13}$$

Compute $h(2D+1)$ and $h(-2D-1)$ to find i and j such that $h(x) = h_i(x)$ for $x \geq 2D+1$ and $h(x) = h_j(x)$ for $x \leq -2D-1$. Then $h(x)$ has no finite minimum if and only if $a_i = 0$ and $b_i < 0$, or $a_j = 0$ and $b_j > 0$.

Finally we comment that the above analysis can also be applied to certain algorithms solving the weighted Euclidean 1-center problem: Given n points (x_i, y_i), $i = 1, \dots, n$, together with positive weights $w_1, \dots, w_n$, find a point (x, y) minimizing $\text{Maximum}_{i=1,\dots,n}\{w_i((x - x_i)^2 + (y - y_i)^2)^{1/2}\}$.

The algorithms presented in [19] use only comparisons between algebraic numbers. A careful examination shows that those numbers are, in fact, roots of quadratics whose integral coefficients are obtained in constant time. Therefore, comparisons can be performed exactly as explained above, and the supposition that the square root operation takes constant time can be removed. In polynomial time (depending only on n), these modified algorithms will find two quadratics having x and y, respectively, as their roots.

6. On root multiplicity and root finding procedures

In Sections 3–4 we introduced a scheme that represents the solutions to the quadratic fractional problem and the ratio game as the unique roots of some polynomials over specified intervals. For example, let $P(\lambda)$ and I denote the polynomial and the interval, respectively, that are generated for the quadratic fractional problem. Also let λ^* be the unique root of $P(\lambda)$ in I. (Let k denote the degree of $P(\lambda)$ and let h denote its height.)

If one wishes to apply classical iterative procedures converging to λ^*, it is advantageous to know the multiplicity of λ^*. For example, Newton Raphson method, which is not directly applicable for a multiple root, can easily be modified (when the multiplicity is known) such that the quadratic rate is not affected [23, Chapter 8]. Let us now demonstrate how to compute the multiplicity of λ^*.

Let $P^i(\lambda)$, $i = 0, 1, \dots, k$, denote the i-th derivative of $P(\lambda)$, and suppose that $P^i(\lambda)$, for all i, have no roots in I, but possibly λ^* itself. Using Sturm sequences [9, 25], we can determine whether $P^i(\lambda^*) = 0$, $i = 0, 1, \dots, k$, and the multiplicity of λ^* as a root of $P(\lambda)$ can then be determined. Therefore, it is sufficient to show how the interval I can be reduced to a subinterval I' (containing λ^*), such that no $P^i(\lambda)$ has a root in I' other than (possibly) λ^* itself.

Recall that k and h denote the degree and height of $P(\lambda)$ respectively. Then the height of $P^i(\lambda)$, $i=0,1,\ldots,k$, is clearly bounded above by $k!h$. Define T' to be the set of all real roots of nonzero integral polynomials with degree $\leqslant k$, and height $\leqslant k!h$. Let Δ' be a lower bound on the distance between distinct members of T' (see Theorem 2). If we proceed with the binary search for λ^* over I until the length of the remaining interval, I', is at most $\Delta'/2$, then I' will certainly possess the desired property. By imitating the proof of Theorem 3 we can show that Δ' can be replaced by an expression of the form $2^{-(5k^3+kL)}$, where $L=\mathrm{O}(\log h)$. An analysis similar to that of Section 3 will then prove that the reduction of I to I' can be done in polynomial time.

7. Final remarks

(1) It was mentioned in the Introduction that it is possible to sharpen the representation of an algebraic solution and ask for an integral polynomial with a minimal degree. Such a minimal polynomial can be obtained (in polynomial time), by factoring the integral polynomial $P(\lambda)$, derived above, using the algorithm in [16].

(2) The general fractional programming (or parametric) model discussed in [18] is not applicable to our optimization problems. It is limited to problems with rational solutions, and the only operations which are permissible on the parametric data are additions and comparisons. For example, at this point in time, the model in [18] does not provide a polynomial scheme for finding a root of a parametric linear program (e.g. (11)) in which the entries of the matrix depend on the parameter. This is due to the fact that the ellipsoidal algorithms (which are the only known polynomial schemes for linear programming), use divisions and multiplications. For this reason the simplex method is also not applicable. Therefore, these algorithms can not be used as master algorithms in [18] to yield polynomial schemes for finding a root of the parametric version of the problem.

(3) In a sequel paper we will focus on other optimization problems with algebraic solutions, e.g., geometric programming problems, stochastic games and median location problems.

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