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AN APPLICATION OF Z-MATRICES TO A CLASS OF RESOURCE ALLOCATION PROBLEMS*

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This study focuses on allocation problems that have some of their constraints defined in terms of Leontief input-output matrices, known as Z-matrices. A few properties of these matrices are discussed and then applied to achieve a possible reduction in the dimensionality of the resource allocation models. An allocation problem of the above nature is the subject of the recent work of Luss and Gupta [7], who were concerned about optimal allocation of marketing efforts among substitutional products distributed in different sales territories. The reduction procedure is then applied to their model to yield several extensions.

Introduction

In this study we discuss a few properties of Z-matrices and then apply them to achieve a possible reduction in the dimensionality of a certain class of resource allocation models. A problem of this nature, which has motivated this study, is the recent work of Luss and Gupta [7], who were concerned about optimal allocation of marketing efforts among substitutional products distributed in different sales territories.

Square matrices with nonpositive off-diagonal elements, known as Z-matrices, have been studied extensively in the literature in both applied and theoretical aspects. To name a few we mention the pioneering work of Leontief [6] who applied the class of Z-matrices and the generalization, by now recognized as Leontief matrices, to interindustry input-output models. Dantzig [3], studying linear programs, defined by block diagonal and triangular Leontief matrices, discovered that their special structure leads to a special computational procedure which is more efficient than the ordinary generalized simplex method. For further applications in the context of input-output studies the reader is referred to Gale [5]. Properties and characterizations of Z-matrices as well as Leontief matrices have been provided by many researchers; most of the known results appear in the survey by Fiedler and Ptak [4]. For more recent results, primarily in the area of mathematical programming and complementarity theory, we refer to Veinott [11], [12] and Cottle and Veinott [2]. Nonlinear extensions of Z-matrices are studied by Tamir [9], [10] and Sandberg [8].

This study focuses on allocation problems that have some of their constraints defined in terms of a Z-matrix and whose objective functions are increasing (isotone) in each of their variables. Specifically, k resources are to be allocated among n activities, $(x_1, x_2, \ldots, x_n)$. The net "effectiveness," z_i, of the ith activity on the criterion function depends linearly on all activity levels. Using the following matrix notation, $z = (z_1, \ldots, z_n)^T$, $x = (x_1, \ldots, x_n)^T$ we consider the following allocation model:

$$\begin{aligned} &\text{Maximize} \quad f(z) \\ &\text{Subject to:} \quad g_i(x) \leqslant c_i, \quad i = 1, \ldots, k, \\ &\qquad\qquad\qquad z = Ax + \mathbf{b}, \quad z \geqslant 0,\ x \geqslant 0, \end{aligned} \tag{1}$$

where A is a square Z-matrix and $\mathbf{b}$ is a given nonpositive vector of dimension n. f, the objective function, is an isotone (increasing) scalar function in the variables

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$(z_1, \ldots, z_n)$ and $g_i(x) \leqslant c_i$, $i = 1, \ldots, k$, express the constraint on the ith resource. We assume that the resource requirements increase with the activity level, i.e. $g_i(x)$ is nondecreasing with $x_1, x_2, \ldots, x_n$.

We start with a description of a general procedure to reduce the dimensionality (i.e. number of variables and constraints) of the above allocation problem. In particular, it will be shown that it is sufficient to consider only those variables of the problem associated with a principal submatrix of the Z-matrix A, having all its principal minors positive. We construct an equivalent formulation of (1) where the x variables as well as the subset of the z variables not associated with the above principal submatrix are omitted. In the next section we discuss a few properties of Z-matrices and present the reduction theorem. The reduction scheme is then introduced. In the third section we elaborate on an extension of the resource allocation model studied by Luss and Gupta [7].

Z-Matrices and the Reduction of the Feasible Set

A square matrix with nonpositive off-diagonal elements is called a Z-matrix. Furthermore if all the principal minors of a Z-matrix ($2^n - 1$ in number) are positive, then the matrix is said to be an M-matrix (M for Minkowski). The following notation will be used throughout. Given an $m \times n$ matrix $A = (a_{ij})$, $A \geqslant 0$ ($A > 0$), if $a_{ij} \geqslant 0$ ($a_{ij} > 0$) for all i, j, $i = 1, \ldots, m$, $j = 1, \ldots, n$.

Let A be an $n \times n$ Z-matrix, let $\mathbf{b}$ be a nonpositive vector of dimension n and consider the polyhedral set defined by

$$z = Ax + \mathbf{b}, \qquad x \geqslant 0, z \geqslant 0. \tag{2}$$

We present in this section a procedure to reduce the dimensionality of the feasible region, based on the properties of the matrix A.

We start by citing a result due to Gale [5, p. 298] and Veinott [11], whose modification to the case of a Z-matrix is the following.

THEOREM 1. *If A is a Z-matrix, then one can partition A, after suitably permuting its rows and columns, so that*

$$A = \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} \tag{3}$$

where A_1 is a square M-matrix, A_3 is a nonpositive matrix and A_2 is a Z-matrix satisfying $x \geqslant 0$ and $A_2 x \geqslant 0$ implies $A_2 x = 0$.

(It is understood, of course, that some of the submatrices in (3) may have no rows and/or no columns, in which case they are omitted from (3).)

Supposing that A is partitioned as in (3) and given a nonnegative vector $\mathbf{q}$, Veinott [11] considered the following set

$$X(\mathbf{q}) = \left\{ x \mid \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} \mathbf{q}^1 \\ \mathbf{q}^2 \end{pmatrix} \quad ; x = (x^1, x^2)^T \geqslant 0 \right\}$$

and observed that $X(\mathbf{q})$ is not empty if and only if $\mathbf{q}^2 = 0$. It is also shown in [11] that if $x = (x^1, x^2)^T$ is an extreme point of $X(\mathbf{q})$, then $x^2 = 0$.

Considering the polyhedral set defined by (2), we rewrite it as

$$\begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} z^1 - \mathbf{b}^1 \\ z^2 - \mathbf{b}^2 \end{pmatrix} \geqslant 0.$$

Thus the above results imply that a feasible solution exists if and only if $z^2 = \mathbf{b}^2 = 0$. Further it will now be demonstrated that the monotonicity assumption on the resource requirement functions $g_i(x)$, $i = 1, \ldots, k$, implies that we may also set $x^2 = 0$ in (1).

To see this, define the region $X^+(\mathbf{q}) = \{x \mid Ax \geqslant \mathbf{q}, x \geqslant 0\}$ where A is the above Z-matrix and $\mathbf{q} = z - \mathbf{b} \geqslant 0$. It is shown in [2], [10] that if $X^+(\mathbf{q})$ is nonempty it contains a least element, $\bar{x}$, (i.e. $\bar{x} \in X^+(\mathbf{q})$ and $\bar{x} \leqslant y$ for all $y \in X^+(\mathbf{q})$) and $\bar{x}(A\bar{x} - \mathbf{q}) = 0$. It is then easily verified that the Z-property of A and the nonnegativity of $\mathbf{q}$ imply that $A\bar{x} = \mathbf{q}$. Furthermore, the minimality property of $\bar{x}$ yields $\bar{x}^2 = 0$. Now, let (x, z) be a feasible solution to (1), then the monotonicity property of $g_i(x)$, $i = 1, \ldots, k$, allows one to focus only on solutions of the form $(\bar{x}(z), z)$ where $\bar{x}(z)$ is the least element of $\{x \mid Ax \geqslant z - \mathbf{b}, x \geqslant 0\}$.

Observe that if A is partitioned as in (3), then for a given $z \geqslant 0$ $\bar{x}(z) = (A_1^{-1}(z^1 - \mathbf{b}^1), 0)^T$, provided $z^2 = \mathbf{b}^2 = 0$ and A_1 is not vacuous. (If A_1 is vacuous $\bar{x}(z)$ is the zero vector.) Assuming now that A_1 is known, then (1) can be reduced to the following equivalent program

$$\begin{aligned} &\text{Maximize} && f\left(\begin{pmatrix} z^1 \\ o \end{pmatrix}\right) \\ &\text{Subject to:} && g_i\left(\begin{pmatrix} A_1^{-1}(z^1 - \mathbf{b}^1) \\ 0 \end{pmatrix}\right) \leqslant c_i, \qquad i = 1, \ldots, k, \\ & && z^1 \geqslant 0. \end{aligned} \tag{4}$$

In order to reduce (1) to its equivalent form (4) one has to find A_1 which is some principal submatrix of the original matrix A. (Note that the representation (3) assumes possible permutations of rows and columns.)

To find A_1 we assume that A is given in its original form. Then following Gale [5, p. 298] and Veinott [11] we observe that A_1 is the principal submatrix of A determined by those indices $i \in \{1, \ldots, n\}$ for which $\{x \mid Ax \geqslant e_i, x \geqslant 0\}$ is not empty (e_i is the ith unit vector).

Neither Gale nor Veinott elaborate on a procedure to find the matrix A_1. Of course, it follows from their works that one can apply any linear programming algorithm to test whether the sets $X(e_i)$, $i = 1, \ldots, n$, are nonempty.

We focus now on the sets $X(e_i)$ and $X^+(e_i)$ and show that the nonemptiness of these sets is verified in at most n pivot operations of Phase I of the simplex procedure, provided that A is of order n.

We start with the following lemma.

Lemma. *Let A be a Z-matrix and $\mathbf{q} \geqslant 0$, then if $X^+(\mathbf{q})$ is nonempty it contains a least element which is the only extreme point of $X^+(\mathbf{q})$ and $X(\mathbf{q})$.*

Proof. The existence of a least element, $\bar{x}$, satisfying $A\bar{x} = \mathbf{q}$ is discussed above. Also note that $\bar{x}$ is an extreme point of both $X(\mathbf{q})$ and $X^+(\mathbf{q})$. Let $y \neq \bar{x}$ be in $X^+(\mathbf{q})$. Then the minimality of $\bar{x}$ yields $2y - \bar{x} \geqslant 0$. Also $A(2y - \bar{x}) \geqslant 2\mathbf{q} - \mathbf{q} = \mathbf{q}$. Hence, $y = \frac{1}{2}(2y - \bar{x}) + \frac{1}{2}\bar{x}$ implies that y is not an extreme point. A similar argument shows that $\bar{x}$ is also the unique extreme point of $X(\mathbf{q})$.

To test for the feasibility of $X(\mathbf{q})$, $\mathbf{q} \geqslant 0$, the system of equations $Ax = \mathbf{q}$ is augmented with the nonnegative artificial variables ω to yield $Ax + \omega = \mathbf{q}$. Using the ω variables as the starting feasible basis, the feasibility of $X(\mathbf{q})$ is examined by minimizing the sum of the artificial variables on the set $\{(x, \omega) \mid Ax + \omega = \mathbf{q}, x \geqslant 0, \omega \geqslant 0\}$. To prove that Phase I yields the solution in at most n pivot operations it is sufficient to show that if an x variable enters the basis it will remain basic and not be pivoted out.

Recalling that an artificial variable is omitted from the simplex tableau when it becomes nonbasic, it is sufficient to prove that no two x variables can have positive coefficients in the same row of the tableau. The latter is clearly satisfied since A is a Z-matrix and so is any matrix obtained from A by a pivot operation. Hence, the proof is complete.

We also note that Phase I will also test the feasibility of $X^+(\mathbf{q})$ in at most n pivot operations. This is implied by observing that the slack variables corresponding to $Ax \geqslant \mathbf{q}$ will always be associated with nonpositive columns and thus will not enter the basis.

Finally we comment that by pivoting out the artificial variables from the optimal solution in the presence of degeneracy, Phase I finds an extreme point of $X(\mathbf{q})$ (and $X^+(\mathbf{q})$) provided nonemptiness. By the lemma this extreme point is the least element.

The above discussion is now summarized.

THEOREM 2. *If A is an $n \times n$ Z-matrix and $\mathbf{q} \geqslant 0$ then Phase I of the simplex method tests the feasibility of $X(\mathbf{q})$ and $X^+(\mathbf{q})$ in at most n pivot operations and yields the least element if feasibility exists.*

Instead of using the regular Phase I of the simplex procedure to test the nonemptiness of $X(e_i)$ we suggest using the algorithm, due to Chandrasekaran [1], for finding a complementary solution to the linear complementarity problem defined by a Z-matrix. (The linear complementarity problem defined by a square matrix A and a vector $\mathbf{q}$ is to find $x \geqslant 0$ such that $Ax + \mathbf{q} \geqslant 0$ and $x(Ax + \mathbf{q}) = 0$.)

As shown in [10], Chandrasekaran's algorithm, when applied to a Z-matrix A, yields the least element of the feasible region, which is also a complementary solution. Hence, it can be used to verify whether $\{x \mid Ax - e_i \geqslant 0, x \geqslant 0\}$ is nonempty.

If A is an $n \times n$ Z-matrix Chandrasekaran's scheme finds the least element by solving a sequence of linear equalities. In fact, it requires (at most) the inversion of n nested principal submatrices. Thus this algorithm is of the polynomial type (i.e. number of elementary operations is polynomial in n).

In fact, upon examination of Chandrasekaran's procedure when applied to $X^+(\mathbf{q})$, $\mathbf{q} \geqslant 0$, one concludes that it is simply a modified Phase I procedure. Instead of introducing a single basic variable at each iteration, as done by the regular Phase I method, Chandrasekaran's scheme allows simultaneous pivoting on a set of variables. Thus we suggest applying the latter for determining the M-matrix A_1 in the partition (3).

A final comment is in order. Chandrasekaran's algorithm finds the least element of $X^+(\mathbf{q})$ in at most n pivots for any $\mathbf{q}$ (without any sign restriction on its components). But this property is not satisfied by the regular Phase I procedure as illustrated by the following example:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3/4 & 0 \\ -1 & -1/2 & 1/8 \end{bmatrix} \quad \text{and} \quad \mathbf{q} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}.$$

An Application to a Marketing Effort Allocation Model

This section extends a model of allocation of marketing effort among substitutional (or neutral) products, recently discussed by Luss and Gupta [7]. In particular, it is shown that the assumptions on the substitutionality factors between the different products can be relaxed without affecting the solution procedures.

We adopt the notation used in [7].

Denote: x_{ij}: marketing effort allocated for project j in territory i, $j = 1, 2, \ldots, P$, $i = 1, 2, \ldots, N$, where N denotes the number of territories and P the number of products.

B: total budget available.

R: total sales.

It is assumed that the sales of each product depend on the marketing effort spent on all the products in the same territory. Since the products are assumed to be substitutional (or neutral) the effective effort for product j in territory i will be defined by the following linear relationship:

$$z_{ij} = x_{ij} - \sum_{\substack{k=1,\\ k \neq j}}^{P} h_{ijk} x_{ik} + b_{ij}$$

where h_{ijk} are nonnegative substitutionality factors and b_{ij} is a nonpositive parameter indicating the exogenous substitutionality effect. The latter can be interpreted as the substitutionality effect introduced by competitors promoting products in the same territory.

If $Q_{ij}(z_{ij})$ denotes the sales response function of product j in territory i, then the multiterritory-multiproduct allocation model is as follows:

$$\begin{aligned} \operatorname*{Max}_{x_{ij}} R = \quad & \sum_{i=1}^{N} \sum_{j=1}^{P} Q_{ij}(z_{ij}) \\ \text{Subject to} \quad & \sum_{i=1}^{N} \sum_{j=1}^{P} x_{ij} \leqslant B, \\ & z_{ij} = x_{ij} - \sum_{k=1,\, k \neq j}^{P} h_{ijk} x_{ij} + b_{ij}, \\ & x_{ij} \geqslant 0,\ z_{ij} \geqslant 0, \qquad i = 1, 2, \ldots, N, j = 1, 2, \ldots, P. \end{aligned} \tag{5}$$

The sales response functions, $Q_{ij}(z_{ij})$, are assumed to be increasing with z_{ij}. Note that this model falls within the framework of the general allocation model (1), to which the reduction procedure is applicable. While dealing with techniques to find the optimal allocation, Luss and Gupta concentrated on the special case where only two substitutional products were available. To present their solution method for this case they imposed some assumptions on the substitutionality factors, h_{ijk}. It seems that their assumptions eliminated some possible economic situations, as well as technical difficulties which may arise in dealing with a general but realistic set up.

Here we consider a more general model by relaxing some of their assumptions as well as by introducing an exogenous substitutionality effect. Using the results on Z-matrices presented in the preceding section, it will be shown that with no restrictions on the nonnegative factors h_{ijk}, the nonnegative variables x_{ij} can be eliminated from the allocation problem, possibly together with some other constraints, to yield a separable optimization problem in the remaining nonnegative variables z_{ij}. The reduction procedure that we use will also indicate whether any marketing effort should be made.

We start by observing that the linear equations in (5) can be written as a block diagonal Z-matrix of order PN. There are N block matrices (one for each territory), each of order P. Writing $z_i = (z_{i1}, \ldots, z_{iP})^T$, $x_i = (x_{i1}, \ldots, x_{iP})^T$, $b_i = (b_{i1}, \ldots, b_{iP})^T$ and

$$A_i = \begin{bmatrix} 1 & -h_{i12} & \cdots & & -h_{i1P} \\ -h_{i21} & \ddots & & & -h_{i2P} \\ & & 1 & & \\ \vdots & & & \ddots & \vdots \\ -h_{iP1} & \cdots & & \cdots & 1 \end{bmatrix}$$

the block corresponding to the ith territory is given by $z_i = A_i x_i + b_i$ where A_i is a Z-matrix of order P. We concentrate now on the ith territory and show how to eliminate the variables $x_i = (x_{i1}, \ldots, x_{iP})$ (and the corresponding nonnegativity constraints) from the allocation problem. If all the principal minors of A_i are positive then A_i^{-1} is nonnegative (see [4]) and the variables x_i can be eliminated using $x_i = A_i^{-1}(z_i - b_i)$. If A_i is either singular or $A_i^{-1} \ngeqslant 0$ we apply the reduction procedure. If after applying this scheme the set of remaining variables is empty, (i.e. if the corresponding M-submatrix in the partition (3) is empty) then we conclude that no marketing effort should be spent in the ith territory under the current circumstances. If this is not the case then the remaining variables, say I_i where $|I_i| \leqslant P$, are associated with a principal submatrix of A_i, B_i of order $|I_i|$, that has positive minors. Denote the remaining variables and constants by x_i^1, z_i^1 and b_i^1 respectively. We then can reduce (5) to:

$$\text{Maximize} \quad \sum_{i=1}^{N} \sum_{j \in I_i} Q_{ij}(z_{ij})$$

$$\text{Subject to:} \quad \sum_{i=1}^{N} e^i B_i^{-1}(z_i^1 - b_i^1) \leqslant B, \tag{6}$$

$$z_i^1 \geqslant 0, \qquad i = 1, \ldots, N,$$

where I_i is the set of variables not eliminated by the reduction procedure in territory i, and e^i is the vector of dimension $|I_i|$ all of whose components equal 1. Efficient solution procedures for (6) are discussed in [7].

Finally several remarks are in order. First we note that in the general set up (1), one can also consider substitutionality between products allocated in different territories. In this case the block diagonal structure is destroyed but the reduction procedure is still applicable to the matrix A.

We have observed that if for all territories $A_i^{-1} \geqslant 0$, then we simply eliminate all variables x_i by using the relations $x_i = A_i^{-1}(z_i - b_i)$. Sufficient conditions on h_{ijk} ensuring the nonnegativity of A_i^{-1} for all territories for the case of 2 and 3 products are provided by Luss and Gupta [7]. Characterizations of M-matrices given in [4] can be used to extend their result to the general case. An economic interpretation of the mathematical conditions also follows. Specifically, we show that a sufficient condition for $A_i^{-1} \geqslant 0$ is that for each product j the marginal cumulative substitutionality effects of the rest of the products, $\sum_{k \neq j} h_{ijk}$, is smaller than the direct marginal effect of the jth product on the effectiveness measure of this product. (In this model this marginal effect is normalized to 1.)

To show that

$$1 - \sum_{\substack{k=1,\\ k \neq j}}^{P} h_{ijk} > 0 \quad \text{for all } j, j = 1, \ldots, P, \tag{7}$$

implies the nonnegativity of A_i^{-1}, we let $\mathbf{x}$ be a vector of dimension P all of whose components equal 1. Then (7) yields $A_i\mathbf{x} > 0$. As shown in [4] the latter condition implies $A_i^{-1} \geqslant 0$. Observe that (7) is satisfied if $h_{ijk} < 1/(P-1)$ for all $j, k = 1, \ldots, P$.

Finally we note that if there are no neutral products in the ith territory (i.e. all products are substitutional and $h_{ijk} > 0$), then either $A_i^{-1} \geqslant 0$, or else no effort should be spent in that territory. This is observed from the partition (3), which implies that if A contains no zero elements, then either A_1 or A_2 is vacuous.

We close with a comment concerning the linearity of the constraints in (5). It seems that the linearity assumption used in defining the effectivity measures, z_{ij}, might be found to be very restrictive in the effort to obtain useful results. Recently, (see [8], [9], [10]), experimental nonlinear extensions of the Leontief input-output system have been set up. These models have a special structure which can be exploited to obtain efficient solution techniques. Further, many properties of the linear model are satisfied by the nonlinear model as well. It is believed that the proposed models will be more applicable to real situations.[1]

[1] The author would like to thank a referee for calling his attention to the partitioning result of Gale [5] and Veinott [11].

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