ON THE INTEGRALITY OF AN EXTREME SOLUTION TO PLUPERFECT GRAPH AND BALANCED SYSTEMS

R. CHANDRASEKARAN

University of Texas at Dallas, Box 688, Richardson, TX 75080, USA

A. TAMIR

Department of Statistics, Tel Aviv University, Tel Aviv, Israel

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Let \( A \) be a nonnegative integer matrix, and let \( v \) denote the vector all of whose components are equal to 1. The pluperfect graph theorem states that if for all integer vectors \( b \) the optimal objective value of the linear program \( \min (c'x | Ax \geq b, x \geq 0) \) is integer, then those linear programs possess optimal integer solutions. We strengthen this theorem and show that any leximaximal optimal solution to the above linear program (under any arbitrary ordering of the variables) is integral and an extreme point of \( (A|Ax \geq b, x \geq 0) \). We note that this extremality property of integer solutions is also shared by covering as well as packing problems defined by a balanced matrix \( A \).

Perfect graphs*, balanced matrices, integral extreme points

The pair \((A, c)\) of an \((m \times n)\) integer matrix \(A\) and an integral \(v\)-vector \(c\) is said to be totally dual integral (TDI) if

\[
\min (c'x | Ax \geq b, x \geq 0) = \min (c'x | Ax \geq b, x \geq 0, x \text{ integer})
\]

for all integer \(b\) for which the linear program has optimal solutions.

Clearly, if \((A, c)\) is TDI, then the optimal value of the objective of the linear program is integral for all integer \(b\) for which it exists. The converse statement is not always true.

Let

\[
A = \begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}
\]

and \(v = (2, 2, 2)\),

then the optimal objective value of \(\min (c'x | Ax \geq b, x \geq 0)\) is integral for all integer \(b\). However, \((A, c)\) is not TDI (Try \(b = (1, 1, 1)\)).

If we restrict our attention to nonnegative matrices \(A\) and set \(v = (1, \ldots, 1)\) then the above converse statement is true and is known as the pluperfect graph theorem [1,2].

Theorem 1. Let \(A\) be an \(m \times n\) nonnegative integral matrix with no zero rows. For any integer \(b\) define

\[
f(b) = \min \left( \sum_{j=1}^{n} x_j | Ax \geq b, x \geq 0 \right).
\]

Suppose that \(f(b)\) is integer for all integer vectors \(b\). Then for any integer vector \(b\) there exists an integer vector \(x^* = x^*(b)\) such that \(Ax^* \geq b\), \(x^* \geq 0\) and \(f(b) = \sum_{j=1}^{n} x_j^0\).

In this work we strengthen the above theorem and show that any leximaximal optimal solution to the linear program (under any arbitrary ordering of the variables) is integral and is an extreme point of \((\{x | Ax \geq b, x \geq 0\}\).

We discuss certain algorithmic implications of this result and show how to find such an optimal integer extreme point.

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Finally we note that this extremality property of integer solutions holds also for covering as well as packing problems defined by a balanced matrix $A$.

To facilitate the discussion let $\Pi$ denote a permutation of $(1, 2, \ldots, n)$. Given two vectors $x, y \in \mathbb{B}$ we say that $x$ is $\Pi$-lexicographic larger than $y$ if, for some index $i$, $x_{\Pi(i)} > y_{\Pi(i)}$ and $x_{\Pi(j)} = y_{\Pi(j)}$ for all $j < i$.

$x$ is a $\Pi$-lexicomaximal vector in a set $S$ if $x$ is in $S$ and $x$ is $\Pi$-lexicographic larger than any $y \neq x$, $y$ in $S$. $x$ is $\Pi$-leximinimal vector in $S$ if $-x$ is $\Pi$-lexicocomaximal in the set $-S$.

We will be concerned here with the case where $S$ is the set of optimal solutions to a linear program. In this case we will say that $x$ is a $\Pi$-lexicomaximal (minimal) optimal solution.

It is well known that a $\Pi$-lexicomaximal (minimal) optimal solution is an extreme point of the feasible domain of the linear program.

Throughout the paper $A_j$ will denote the $j$-th column of a matrix $A$. $[a]$ (or $[a]$) will denote the smallest (largest) integer which is not smaller (larger) than $a$.

**Lemma 2.** Let $(A, c)$ be a pair of an $(m \times n)$ nonnegative integral matrix $A$ and an $(n \times 1)$ nonnegative integral vector $c$. Let $K = \{k|c_k = 1\}$.

Suppose that for all integer $b$, $f(b)$, defined by

$$ f(b) = \min \{c^T x | Ax \geq b, x \geq 0\}, $$

is integer. ($A$ is assumed to contain no zero rows so that $f(b)$ is well defined.)

Let $\Pi$ be a permutation such that $\Pi(i) \in K$ for $i = 1, \ldots, |K|$. Then $x^{\Pi}$ is the $\Pi$-lexicomaximal optimal solution to

$$ \min \{c^T x | Ax \geq b, x \geq 0\}. $$

Then $x^{\Pi}$ is an extreme point of $(x | Ax \geq b, x \geq 0)$ and $x_k^{\Pi}$ is integer for all $k \in K$.

**Proof.** As noted above for any permutation $\Pi$, a $\Pi$-lexicomaximal optimal solution to a linear program is an extreme solution.

Without loss of generality suppose that $K = \{1, 2, \ldots, t\}$ and $\Pi(k) = k, k = 1, 2, \ldots, t$. Furthermore, assume that for some $r, r < t$, $x^r$ is not integral, while $x^{t-1}$ is integral for $1 \leq j \leq r - 1$.

Define $x^0$ as follows:

$$ x^0 = 0, \quad 1 \leq j \leq r, \quad x^0 = x^r, \quad r + 1 \leq j \leq n. $$

Now clearly satisfies $A x^0 \geq b, y^0 \geq 0$, where

$$ b = b - \sum_{j=1}^{r-1} x^0 A_{., j} - x^r A_{., r}. $$

Furthermore

$$ f(b) \leq c^{r-1} \sum_{j=1}^{r-1} x^0 - x^r - (c^{r-1} \sum_{j=1}^{r-1} x^0 - [x^0] + 1. $$

Since $f(b)$ as well as

$$ c^{r-1} \sum_{j=1}^{r-1} x^0 - [x^0] + 1 $$

are both integer,

$$ f(b) \leq c^{r-1} \sum_{j=1}^{r-1} x^0 - [x^0]. $$

(2)

Let $y^*$ be an optimal solution yielding $f(b)$. Define $z^*$ by

$$ z^* = \begin{cases} x^0 + y^*, & 1 \leq j \leq r - 1, \\ x^r, & j = r. \end{cases} $$

Then $z^* s$ satisfies $A z^* \geq b$ and $z^0 \geq 0$. By (2),

$$ f(b) \leq c^{r-1} - \sum_{j=1}^{r-1} x^0 - [x^0]. $$

Hence $c^{r-1} = f(b)$, and $z^0$ is an optimal solution. But $z^0$ is $\Pi$-lexicographic larger than $x$ — thus contradicting the supposition that $x^r$ is not integral.

The above lemma implies the following strengthened version of the pluperfect graph theorem (Theorem 1).

**Theorem 3.** Let $A$ be an $(m \times n)$ nonnegative integral matrix with no zero rows. For any integer $b$

$$ f(b) = \min \{c^T x | Ax \geq b, x \geq 0\}. $$

Suppose that $\Pi$ is integer for all integer $b$. Consider an integer vector $b$. Let $\Pi$ be a permutation of $(1, \ldots, n)$ and let $x_{\Pi}$ be a $\Pi$-lexicomaximal solution.
optimal solution to

\[
\min \left\{ \sum_{j=1}^{n} x_j | A x \geq b, x \geq 0 \right\}.
\]

Then \( x_{ii} \) is an integral extreme point of \( \{ x | A x \geq b, x \geq 0 \} \).

Corollary 4. Let \( A \) satisfy the conditions of Theorem 3. Given an integer vector \( b \), the solution to the integer program

\[
\min \left\{ \sum_{j=1}^{n} x_j | A x \geq b, x \geq 0, x \text{ integer} \right\}
\]

is unique if and only if the solution to the linear program

\[
\min \left\{ \sum_{j=1}^{n} x_j | A x \geq b, x \geq 0 \right\}
\]

is unique.

Proof. Let \( x^0 \) denote the lexicomaximal optimum solution with respect to the identity permutation. If the linear program has more than one optimal solution, there exists an optimal solution \( x^0 \) such that \( y_0 > x^0 \) for some index \( r \). Let \( \Pi \) be a permutation where \( \Pi(1) = r \). The \( \Pi \)-lexicomaximal optimal solution \( x^\Pi \) satisfies \( x^\Pi \neq x^0 \).

We note that any matrix \( A \) that satisfies the conditions of the above theorem must be a 0–1 matrix. Moreover, \( A \) is the node clique incidence matrix of some \( m \)-vertex perfect graph, where rows represent the nodes and columns correspond to cliques \([1,2]\). The problem of finding an integer vector \( x \) yielding \( f(x) \) can then be viewed as the weighted clique cover problem on a perfect graph. The latter problem is solved polynomially in the seminal paper [4] by Grötschel, Lovász, and Schrijver, where the input of the problem consists only of the \((n \times m)\) node vs. node incidence matrix and the vector \( b \).

The algorithm suggested in [4] for solving the weighted clique cover can be schematically described as follows:

0. Set \( z_0 = 0, j = 1, \ldots, n \).
1. Solve the linear program

\[
\min \left\{ \sum_{j=1}^{n} x_j | A x \geq b, x \geq 0 \right\},
\]

obtaining an optimal solution \( x^\ast \).

Let \( J = \{ j | x^\ast_j > 0 \} \).
2. Set \( z_j = z_j + | x^\ast_j | \) for each \( j \in J, b \leftarrow b - \sum_{j \in J} x^\ast_j A_{j} \).
3. Set \( x^\ast_{j'} = 0 \) for each \( j' \notin J \) stop; \( z \) is an optimal integer solution. Otherwise let \( r \) be such that \( x^\ast_r - | x^\ast_r | > 0 \). Go to 4.
4. Set \( z_r = z_r + 1, b \leftarrow b - A_{r} \), and go to 1.

Of course the main contribution of [4] is the polynomial algorithm to solve the linear programs defined in Step 1. As mentioned above, in their algorithm, \( A \) is not listed explicitly, and the optimal vector \( x^\ast \) will have at most \( m \) positive components, i.e., \(|J| \leq m\).

Grötschel et al. [4] use a direct perfect graph argument to validate the above scheme (in particular the rounding up in Step 4).

A different justification can be obtained by Theorem 3 without referring to the underlying perfect graph. The noninteger vector \( y \), defined by

\[
y_j = \begin{cases} x^\ast_j - | x^\ast_j |, & \text{if } j \in J, \\ 0, & \text{if } j \notin J, \end{cases}
\]

optimally solves the linear program with \( b - \sum_{j \in J} x^\ast_j A_{j} \), as the r.h.s. Therefore, \( y \), if \( y \geq 0 \), it follows from Theorem 3 that there exists an integer optimal solution to the latter problem with the \( r \)-th component being at least 1. This validates Step 4 and the above scheme in general.

The above scheme will (polynomially) generate an optimal integer solution, provided that some polynomial routine (in terms of the length of the input used to define the problem) is used in Step 1. However, such a scheme may not generate an optimal integer solution which is also an extreme point. Theorem 3 suggests the following procedure that finds an optimal integer extremum solution.

0. Set \( z_0 = 0, j = 1, \ldots, n \).
1. Solve the linear program

\[
f(b) = \min \left\{ \sum_{j=1}^{n} x_j | A x \geq b, x \geq 0 \right\}.
\]

Let \( r \) be some positive component of an optimal solution, (If none exists, \( z \) is an optimal integer extremum solution).
2. Find the largest integer, \( p \), in \([0, b]\), where \( b = \max_{j} (b_j) \), such that \( f(b) = f(b - pA) + p \).
3. Set \( z_r = p, b \leftarrow b - pA \).
4. Go to 1.
Theorem 3, coupled with the following observation, justifies the validity of this algorithm. Let $z_{i,j}, j = 1, \ldots, q$, be the positive component of $z$ produced at the $j$th iteration of the algorithm. Then, $z$ is a $\Pi$-lexicomaximal optimum solution for any permutation $\Pi$ such that $\Pi(j) = i(j), j = 1, \ldots, q$.

The integer $p$ defined in Step 2, can be obtained by applying a binary search on the integers in $[0, \delta]$. Therefore, $p$ is computable by solving $O(\log \delta)$ linear programs like those defined in Step 1. The algorithm iterates at most $m$ times since an extreme point has at most $m$ positive components.

Thus, we conclude that the above procedure will generate an optimal integer extreme point provided that some polynomial routine is available for solving the linear program in Step 1. (In fact this routine will need to produce only the value $f(b)$ and the index of one positive component of an optimal solution.)

We now demonstrate that the linear program in Theorem 3 can have a noninteger extreme optimal solution even when $b$ is the vector all of whose components are equal to 1.

Example. Let $G$ be the six vertex graph shown in Figure 1. It is a perfect graph since its complement consists of three isolated edges only. The node clique incidence matrix of $G$ is

$$A = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1
\end{bmatrix}$$

\[\text{Fig. 1.}\]

If $b$ is the vector all of whose components are equal to 1, $f(b)$ in Theorem 3 is also attained at the noninteger optimal extreme point $(0, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, 6, 0, 0)$.

We conclude by noting that the above extremality property of integral solutions is also shared by covering and packing problems defined by balanced matrices. We state these results but omit the proofs since they are very similar to that of Lemma 2. These results strengthen the results in [3]. Recall that a 0–1 matrix is balanced if it does not contain a square submatrix of odd order whose column and row sums are all equal to 2.

Theorem 5. Let $A$ be an $(m \times n)$ balanced matrix, and let $b$ and $d$ be integer vectors. Let $\Pi$ be a permutation of $(1, \ldots, n)$.

1. Every $\Pi$-lexicomaximal optimal solution to the linear program

$$\min \left\{ \sum_{j=1}^{n} x_j | Ax \geq b, 0 \leq x \leq d \right\}$$

is integral and an extreme point of $\{x | Ax \geq b, 0 \leq x \leq d\}$.

2. Every $\Pi$-leximinimal optimal solution to the linear program

$$\max \left\{ \sum_{j=1}^{n} x_j | Ax \leq b, 0 \leq x \leq d \right\}$$

is integral and an extreme point of $\{x | Ax \leq b, 0 \leq x \leq d\}$.

References