# On Totally Unimodular Matrices

**A. Tamir**
Tel-Aviv University
Tel-Aviv, Israel

ABSTRACT

*Conditions for a matrix to be totally unimodular, due to Camion, are applied to extend and simplify proofs of other characterizations of total unimodularity.*

A matrix is totally unimodular if every square submatrix has determinant +1, -1, or 0. The concept of total unimodularity has been investigated in relation to electrical networks as well as to combinatorial mathematical programming problems. These matrices have been studied extensively since the discovery of Hoffman and Kruskal [13] who proved that an integral matrix A is totally unimodular if and only if the extreme points of $X^*(A,b) = \{x : Ax \leq b,\ x \geq 0\}$ are integral for all integer b. A much simpler proof of this characterization was given by Veinott and Dantzig [16]. For further contributions in characterizing totally unimodular matrices the interested reader is referred to [7,8,9,10,11,15].

One of the earliest results in this field is due to Camion [4] who characterized totally unimodular matrices in terms of Eulerian matrices, (see [1]). The main purpose of our paper is to indicate and expose the great potential of Camion's result, which seems to have been ignored in several newer works, by applying it to extend and refine theorems reported in a more recent work of Chandrasekaran [5]. It is also demonstrated that Camion's characterization provides a short and more elementary proof for a recently derived sufficient condition for total unimodularity [6].

We start with our derivation of the sufficient condition.

In a recent paper, Commoner [6] provided a sufficient condition for a matrix to be totally unimodular. This condition is based upon a directed bipartite graph obtained from a {1,-1,0}

Networks, 6: 373-382

- valued matrix by associating a node with every row and column, and drawing an edge between a row node and a column node if the entry in that row and column is nonzero. The sign of the entry determines the orientation of the edge.

We first introduce several definitions and a few results of [4,5,6]. Let A be a matrix with entries equal to 1, -1, or 0. A submatrix B of A is said to be Eulerian if $e'B = 0 \pmod 2$ and $Be = 0 \pmod 2$ where e is the summation vector (i.e. all its entries equal to 1) of the appropriate dimension. The following two characterizations are given in [4].

*Theorem 1: A matrix A is totally unimodular if and only if every square Eulerian submatrix is singular.*

*Theorem 2: A matrix A is totally unimodular if and only if for every (square) Eulerian submatrix B $e'Be = 0 \pmod 4$.*

Camion's proofs of the above theorems are based on the following observation due to Gomory and reported in [4].

*Theorem 3: If A is a $\{1,-1,0\}$ - valued matrix which is not totally unimodular, then A has a submatrix of determinant $\pm 2$.*

Simple proofs of Gomory's result (due to Tamir and Truemper) as well as its application in implying other characterizations will appear in a survey paper currently in preparation [14].

To present Commoner's result we introduce the necessary concepts from graph theory. A *net* is a finite directed bipartite graph G, such that any two nodes are connected by at most one arc. An *elementary circuit* C of the given graph is a connected subraph all of whose nodes have degree 2. (The degree of a node is the number of arcs incident to the node.) A *chord* on an elementary circuit C is an arc of G that connects two nodes of C but is not an arc of the subgraph C. An elementary circuit with no chord is called a *minimal* circuit. If I and J are the two parts (sets of nodes) of G (i.e. no two members of I(J) are connected by an arc), then define the incidence matrix A of the net G as follows.

For $i \in I$ and $j \in J$ define $a_{ij}$ to be zero if i and j are not connected, +1 if i and j are connected by an arc directed from i to j (we use the notation $\langle i,j \rangle$) and -1 if the direction is from j to i ($\langle j,i \rangle$). The incidence matrix A is then defined by $A = (a_{ij})$. Notice that the assumption of a net that two nodes are connected by at most one arc ensures that the correspondence between nets and $\{1,-1,0\}$ - valued matrices is well-defined.

Let A be the given incidence matrix of net G. Given an elementary circuit C in the net G we define the sign of C as follows. Let $I_1$ ($J_1$) be the set of nodes of C contained in I(J). Since G is bipartite $|I_1| = |J_1|$ and each node i of $I_1$ is connected to exactly two nodes of $J_1$, k(i) and j(i). The sign of C, $\sigma(C)$, is then defined $\sigma(C) = \prod_{i \in I_1} (-a_{ij(i)} \cdot a_{ik(i)})$. The elementary circuit is *even (odd)* if $\sigma(C)$ is +1 (-1).

We now introduce Commoner's sufficient condition for total unimodularity.

*Theorem 4: If each elementary circuit of a net is even then the incidence matrix of the net is totally unimodular.*

Applying a result due to Camion [4], we give a simple and more elementary proof of Theorem 4. The proof will be based on the following lemma.

*Lemma: Let A be the incidence matrix corresponding to a net all of whose elementary circuits are minimal. Suppose that A is not totally unimodular and let B be a minimal square submatrix of A which is not totally unimodular (i.e. each proper submatrix of B is totally unimodular). Then every column (row) of B contains exactly two nonzero entries, and B is the incidence matrix of an elementary circuit.*

*Proof:* The minimality of B and Theorem 2 imply that B is an Eulerian submatrix. Consider the bipartite subgraph $G_1$ having B as its incidence matrix, then the degree of each node of $G_1$ is even (0 mod 2). We also note that the minimality of B ensures that $G_1$ is connected, since otherwise we would have the contradiction $1 < |\det B| = |\det B_1|\ |\det B_2| \leq 1$ where $B_1$ and $B_2$ are proper submatrices of B. Thus the degree of each node of $G_1$ is even and at least two. Therefore (see [2,p.229]) there exists an Eulerian tour on $G_1$. It is easily seen that the tour can be decomposed into $k \geq 1$ elementary circuits. The assumption that each elementary circuit is minimal implies that the incidence matrix of each elementary circuit of the tour is a submatrix of B. Hence B can be partitioned into k submatrices, $B_1, \ldots, B_k$, of the appropriate dimensions, where any two matrices can overlap only on zero elements of B.

Suppose now that there existed a column (row) of B with more than 2 nonzero entries. This would imply that $G_1$ contains a node with degree greater than two. Therefore the Eulerian tour would consist of more than one circuit and $k \geq 2$.

Observe first that $e'Be = e'B_1e + \ldots + e'B_ke$, where $e$ is a summation vector of the appropriate dimension. Next note that from Theorem 2 $e'B_ie = 0 \pmod 4$ $i = 1, \ldots, k$, since $B_1, \ldots, B_k$ are proper submatrices of B. Thus $e'Be = 0 \pmod 4$ which contradicts the minimality of B.

We can now prove Theorem 4.

Suppose that each elementary circuit of a net is even. It is then easily verified that each elementary circuit is minimal (see Commoner [6]). If the incidence matrix A isn't totally unimodular, there exists a minimal submatrix B which is not totally unimodular. From the Lemma each column (row) of B contains exactly two nonzero entries. Further B is the incidence matrix of a minimal circuit. Hence by relabeling the nodes we assume without loss of generality (since $|\det B|$ is not changed by permuting the rows and columns of B) that

$$B = \begin{bmatrix} a_1 & 0 & \cdots & 0 & b_n \\ b_1 & & & & 0 \\ 0 & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & 0 \\ 0 & \cdots & 0 & b_{n-1} & a_n \end{bmatrix}$$

where $a_i$, $b_i$ are equal to +1 or -1. Therefore

$$\det B = \pi a_i + (-1)^{n+1}\pi b_i = \pi a_i - \pi(-b_i) = \frac{\pi a_i(-b_i) - 1}{\pi(-b_i)}. \quad \text{Thus}$$

$|\det B| > 1$ implies that $\pi a_i(-b_i) = -1$; a contradiction to the evenness of the minimal circuit corresponding to B.

Several comments are in order. First note that the sufficient conditions of Theorem 4 are not necessary (not even for $\{0,1\}$ matrices). This is demonstrated by the following totally unimodular incidence matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

that corresponds to a net, containing an odd elementary circuit. It is shown in [6] that evenness of minimal circuits is required for total unimodularity. But the latter condition is not in general sufficient as shown by the incidence matrix

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & & \\ 1 & & 1 & \\ 1 & & & 1 \end{bmatrix}$$

In fact, when dealing with $\{0,1\}$ matrices, evenness of minimal circuits has been shown to be equivalent to the matrix being balanced. Note that the latter property is weaker than total unimodularity. In view of Hoffman and Kruskal [13] characterization of totally unimodular matrices, balanced matrices, first introduced by Berge [3], are those $m \times n$ $\{0,1\}$ matrices A that satisfy the following.

For every $\{0,1\}$ vector w in $R^m$ and for every $\{0,1\}$ vector b in $R^n$ the linear program $\min\{\sum_{i=1}^{m} y_j : y A \geq b,\ 0 \leq y \leq w\}$ provides an integral solution.

Finally we mention a comment of a referee that Theorem 4 had been communicated to him by Edmonds in 1961.

We next turn to a necessary and sufficient condition for total unimodularity given by Commoner [6,(2.5),(A.9)]. To our knowledge this characterization (given as Theorem 5), was first proved by A. Ghouila-Houri [7]. (Also reported in [2,p.468]). For the sake of completeness we provide a simple proof which is based on Theorem 1.

*Theorem 5: An $m \times n$ matrix $A = (a_{ij})$ is totally unimodular if and only if each set $J \subset \{1,2,\ldots,n\}$ can be divided into two disjoint sets $J_1$ and $J_2$ such that*

$$\left| \sum_{j \in J_1} a_{ij} - \sum_{j \in J_2} a_{ij} \right| \leq 1 \quad \text{for all} \quad 1 \leq i \leq m.$$

*Proof:* An equivalent statement of the theorem is the following:

A is totally unimodular if and only if for each submatrix B there exists a vector $\lambda$ (of the appropriate dimension) all of whose components equal to $\pm 1$ and all the components of $B\lambda$ are 0, +1, or -1.

The condition is sufficient: By Theorem 1 it is sufficient to show that every square Eulerian submatrix of A is singular. Let $B = (b_{ij})$, $i \in I$, $j \in J$ be Eulerian. There exists two disjoint sets $J_1$ and $J_2$ such that

$$t_i = \sum_{J_1} b_{ij} - \sum_{J_2} b_{ij} \text{ is either 0, +1, or -1 for all } i \in I.$$

The singularity of B will follow if we show that $t_i = 0$ for all $i \in I$. Suppose on the contrary that $t_i = \pm 1$ for some $i \in I$, then

$$\sum_J b_{ij} = \sum_{J_1} b_{ij} - \sum_{J_2} b_{ij} + 2\sum_{J_2} b_{ij} = \pm 1 + 2\sum_{J_2} b_{ij},$$

i.e. $\sum_J b_{ij}$ is odd, contrary to the supposition that B is Eulerian.

The necessity is proved in two steps.

(1) Let B be such that $Be = 2\alpha$, where e is the summation vector and $\alpha$ is an integral vector. Using the total unimodularity of B, there exists an integral vector $0 \leq \lambda \leq e$ such that $B\lambda = \alpha$, [13,15]. Clearly the vector $\bar{\lambda} = e - 2\lambda$ has all its components equal to $\pm 1$ and $B\bar{\lambda} = 0$.

(2) Let B be a $(p\times r)$ submatrix of A. Define a $p \times p$ matrix $\bar{B}$ as follows. The $i^{th}$ column of $\bar{B}$ is $e_i$, the $i^{th}$ unit vector, if $(Be)_i = 1 \pmod 2$ and the zero vector otherwise. $[B,\bar{B}]$ is totally unimodular (since B is) and $Be + \bar{B}e = 0 \pmod 2$. Hence, from (1) there exists a $(\lambda_1,\lambda_2)$, all of whose components equal to $\pm 1$ and $B\lambda_1 + \bar{B}\lambda_2 = 0$. The proof is now complete since all the components of $\bar{B}\lambda_2$ are equal to 0, +1, or -1.

In the next section we apply Theorems 1 and 2 to extend and refine the following characterization of totally unimodular matrices due to Chandrasekaran [5].

*Theorem 6: A matrix A is totally unimodular if and only if for every nonsingular submatrix* $B = (b_{ij})$ $i,j = 1, \ldots, n$ *the g.c.d. of* $\sum_j \lambda_j b_{1j}, \sum_j \lambda_j b_{2j}, \ldots, \sum_j \lambda_j b_{nj}$ *is 1 for any* $\lambda_j = 0, \pm 1$, *but not all zero.*

The next result proves that it is sufficient to consider the case where $\lambda_j = 1$, for all j, in the above theorem.

*Theorem 7: Suppose that for every nonsingular submatrix* $B = (b_{ij})$ *of A the g.c.d. of* $\sum_j b_{1j}, \sum_j b_{2j}, \ldots,$ *is 1. Then A is totally unimodular.*

*Proof:* Assume that A isn't totally unimodular. Then by Theorem 1 there exists an Eulerian submatrix B and det B $\neq$ 0. We observe that Be = 0 (mod 2) and det B $\neq$ 0 imply that the g.c.d. of $\sum_j b_{1j}, \sum_j b_{2j}, \ldots,$ is at least 2 - a contradiction to the theorem assumption.

While Theorem 7 strengthens the sufficiency condition of the characterization of Theorem 6, the next result, dealing with unimodular matrices, refines the necessary condition of that theorem. (Note that a square matrix is unimodular if its determinant is equal to ±1.)

*Theorem 8: Let B be a square integer matrix. Then B is unimodular if and only if for each integer vector* $\lambda$ *the g.c.d. of the elements of* $B\lambda$ *is equal to the g.c.d. of the elements of* $\lambda$.

*Proof:* Sufficiency: We first show that det B $\neq$ 0. Supposing that det B = 0 and observing that B is an integer matrix defined on the field of rational numbers we conclude that there exists a nonzero rational vector $\mu$ such that $B\mu = 0$. Thus there exists a nonzero integer vector $\lambda$ and $B\lambda = 0$. This clearly contradicts the assumption on the equality of the two greatest common divisors. Hence, det B $\neq$ 0.

Suppose now that the order of B is n and let $c_i$, $i = 1, \ldots, n$, be the $i^{th}$ column of the matrix adj(B). Then $Bc_i = (\det B)e_i$, where $e_i$ is the $i^{th}$ unit vector in $R^n$. Thus the g.c.d. of the elements of $c_i$ is $|\det B|$. The latter implies that det (adj(B)) is an integer multiple of $(\det B)^n$. Hence, $B(\mathrm{adj}(B)) = (\det B)I$ yields

$$(\det B)^n = \det(B(\mathrm{adj}(B))) = (\det B)(\det B)^n\, t$$

where t is integer. Hence $|\det B| = 1$.

Necessity: Let $k \geq 1$ be the g.c.d. of the components of $\lambda$, i.e. $\lambda = k\alpha$ where $\alpha$ is an integral vector. If B is a matrix whose order is equal to the order of the vector $\lambda$, then $B\lambda = kB\alpha$. $B\alpha$ is integral and thus the g.c.d. of the components of $B\lambda$, g, is at least k. Assuming that B is unimodular and using $B^{-1} = \mathrm{adj}(B)/\det B$ we obtain that $B^{-1}$ is an integral matrix. g is the g.c.d. of the elements of $B\lambda$, i.e. $B\lambda = g \cdot \beta$ where $\beta$ is an integral vector. We then have $\lambda = g\, B^{-1}\, \beta$ which in turn implies that $k \geq g$. Thus $k = g$.

Finally, while observing that Theorem 8 is a theorem about unimodular matrices, we point out that Theorem 6 as well as Theorem 7 are results about totally unimodular matrices. They cannot be extended along the lines of Theorem 8. This is demonstrated by the following example due to Chandrasekaran [5].

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$\det B = 5$, but there exist no $\lambda_j = 0, \pm 1$, not all zero, such that the g.c.d. of the elements of $B\lambda$ is not equal to 1.

REFERENCES

1. Berge, C., *The Theory of Graphs*, Methuen, 1962.

2. Berge, C., *Graphs and Hypergraphs*, North-Holland, Amersterdam, 1973.

3. Berge, C., "Balanced Matrices," *Mathematical Programming*, 2, 1972, pp. 19-31.

4. Camion, P., "Characterization of Totally Unimodular Matrices," *Proc. Amer. Math. Soc.*, 16, 1965.

5. Chandrasekaran, R., "Total Unimodularity of Matrices," *SIAM J. of Appl. Math.*, 17, 1969.

6. Commoner, F. G., "A Sufficient Condition for a Matrix to be Totally Unimodular," *Networks*, 3, 1973, pp. 351-365.

7. Ghouila-Houri, A., "Caracterisation des Matrices Totalement Unimodulaires," *C. R. Acad. Sc. Paris*, 254, 1962, pp. 1192-1194.

8. Heller, I. and C. P. Tompkins, *An Extension of a Theorem of Dantzig's*, *Linear Inequalities and Related Systems*, Princeton University Press, Princeton, New Jersey, 1956, pp. 247-254.

9. Heller, I., "On Linear Programs Equivalent to the Transportation Problem," *SIAM J. of Appl. Math.*, 12, 1964.

10. Heller, I., "On Linear Systems with Integral Valued Solutions," *Pacific J. Math.*, 7, 1957, pp. 1351-1364.

11. Heller, I., "On a Class of Equivalent Systems of Linear Inequalities," *Pacific J. Math.*, 13, 1963, pp. 1209-1227.

12. Hoffman, A. J. and I. Heller, "On Unimodular Matrices," *Pacific J. Math.*, 12, 1962, pp. 1321-1327.

13. Hoffman, A. J. and J. B. Kruskal, *Integral Boundary Points of Convex Polyhedra*, *Linear Inequalities and Related Systems*, Princeton University Press, Princeton, New Jersey, 1956, pp. 223-246.

14. Truemper, K. and R. Chandrasekaran, "Algebraic Characterizations of Totally Unimodular Matrices: A Survey," University of Texas at Dallas, in preparation.

15. Tutte, W., "Matroid Theory," *Rand Report R-448-PR*, The Rand Corporation, Santa Monica, California, 1966.

16. Veinott, A. F. and G. B. Dantzig, "Integral Extreme Points," *SIAM Rev.*, 10, 1968, pp. 371-372.

---

*Part of this research was conducted while the author was at Northwestern University.*

*Paper received November 20, 1974.*