Research Proposal: Infinite Horizon Games

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1 Introduction

Infinite horizon games are multi-player games which are played in infinite stages. At every stage either one of the players or some of them, choose an action in a finite set of actions. The payoff to each player is a measurable function of the infinite history of the players' actions. The purpose of this study is to research several infinite horizon games.

The proposal is organized as follows: In section 2 we present our first result concerning subgame perfect equilibria in stopping games. Section 3 exhibit suggestions for further research of infinite horizon games with perfect information.

2 Subgame Perfect equilibria in stopping games

Stopping games (without simultaneous stopping) are sequential games in which at every stage one of the players is chosen, and decides whether to continue the interaction or stop it, whereby the terminal payoff vector is obtained.

Stopping games were introduced by Dynkin (1969), who studied two-player zero-sum games with bounded payoffs. Dynkin proved the existence of the value and pure ϵ-optimal strategies.

Since the game has perfect information, by Mertens (1987) it follows that every multi-player stopping game has an ϵ-equilibrium. Since the ϵ-equilibrium strategies that were constructed by Mertens (1987) employ threats of punishment, which might be non-credible, subsequent work concentrated on the existence of subgame perfect equilibrium in some spacial classes of stopping games (see Solan and Vieille (2003), Solan (2005)).

The first goal of our study is to extend these results for general stopping game, which defined as follows:

A **stopping game** is given by $\Gamma = (I, \Omega, \mathcal{A}, \mathbf{P}, \mathcal{F}, (i_k)_{k=1}^{\infty}, (a_k)_{k=1}^{\infty}, a_\infty)$ where:

- $I = \{1, ..., n\}$ is a non-empty finite set of players.
- $(\Omega, \mathcal{A}, \mathbf{P})$ is a probability space.
- $\mathcal{F} = (\mathcal{F}_k)_{k=1}^{\infty}$ is a filtration over $(\Omega, \mathcal{A}, \mathbf{P})$, representing the information available to the players at stage k.
- $(i_k)_{k=1}^{\infty}$ and $(a_k)_{k=1}^{\infty}$ are $\mathcal{F}$-adapted processes. $(i_k)_{k=1}^{\infty}$ is an I-valued process, which indicates the player who should decide whether to stop the game or to continue. $(a_k)_{k=1}^{\infty}$ is a $\Re^n$-valued process, which indicates the terminal payoff if player i_k stops.
- $a_\infty \in \Re^n$ is a payoff vector, representing the payoff if no player ever stops [1].

The game is played as follows: An element $\omega \in \Omega$ is chosen according to $\mathbf{P}$. At every stage $k \in \mathbf{N}$ player $i_k(\omega)$ decides whether to stop the game or to continue. If player $i_k(\omega)$ decides to stop, the game terminates with

[1] a_∞ will not be normalized to 0 as could seem natural at this point, because another normalization will be used later.

terminal payoff vector $a_k(\omega)$. If player $i_k(\omega)$ decides to continue, the play continues to stage $k+1$. If the game never terminates, the payoff vector is a_∞.

To save notations, we assume that the players choose actions even after the game terminates.

A **(behavioral) strategy** for player i is a $[0,1]$-valued $\mathcal{F}$-adapted process $\sigma^i = (\sigma^i_k)_{k=1}^\infty$. $\sigma^i_k(\omega)$ is the probability that player i stops at ω when chosen at stage k (provided the game did not terminate before that stage). A **profile** is a vector of strategies, one for each player. We denote by σ^{-i} the vector of strategies of all the players excluding player i.

A **play** is given by ω and an infinite sequence of players' actions, therefore it can be identified by an infinite sequence $(i_1, a_1, b_1, i_2, a_2, b_2, ...)$ which includes the chosen players, their terminal payoffs and chosen actions. Each profile σ induce a distribution $\mathbf{P}_\sigma$ over the set of plays. Denote by $\gamma(\sigma)$ the expected payoff vector under σ.

Whereas the probability space is arbitrary, the filtration does not necessarily include atoms, hence we had to redefine the concept of subgame perfect equilibrium.

For every $F \in \mathcal{A}$, such that $\mathbf{P}(F) > 0$, denote by $\gamma_{|F}(\sigma)$ the conditional expected payoff vector under σ given F occurs.

DEFINITION 2.1 *Let $\epsilon \geq 0$, and $F \in \mathcal{A}$ such that $\mathbf{P}(F) > 0$. A profile σ is an* ϵ-**equilibrium on** F *if for every player $i \in I$ and every strategy $\overline{\sigma}_i$ of player i,*

$$\gamma^i_{|F}(\sigma) \geq \gamma^i_{|F}(\sigma^{-i}, \overline{\sigma}_i) - \epsilon.$$

In particular, σ is an ϵ-equilibrium if and only if σ is an ϵ-equilibrium on Ω.

For every $K \in \mathbf{N}$ define the game that starts at stage K, by

$$\Gamma_{|K} = (I, \Omega, \mathcal{A}, \mathbf{P}, (\mathcal{F}_k)_{k=K}^\infty, (i_k)_{k=K}^\infty, (a_k)_{k=K}^\infty, a_\infty).$$

Every strategy σ^i of player i in Γ induces a strategy $\sigma^i_{|K}$ in $\Gamma_{|K}$, by ignoring the play in the first $K-1$ stages.

Definition 2.2 *Let $\epsilon, \delta \geq 0$. A profile σ is a* **δ-approximate subgame perfect ϵ-equilibrium** *iff, for every $F \in \mathcal{A}$ such that $\mathbf{P}(F) > \delta$, $\sigma_{|K}$ is an ϵ-equilibrium in $\Gamma_{|K}$ on F for every $K \in \mathbf{N}$.*

The main result is: Every stopping game such that $sup_{k\in\mathbf{N}}\|a_k\|_\infty \in L^1(\mathbf{P})$ has a δ-approximate subgame perfect ϵ-equilibrium, for every $0 < \delta, \epsilon$.

Following Solan and Vieille (2003), a stopping game needs not have a subgame perfect 0-equilibrium. However, it is not clear yet, whether this result is tight concerning δ, that is whether there is a stopping game which does not have a 0-approximate subgame perfect ϵ-equilibrium. We intend to go on answer this question.

3 Subgame Perfect equilibria in deterministic infinite horizon games with perfect information

A deterministic infinite horizon game with perfect information is an infinite horizon game in which at every stage one player is chosen and only that player chooses an action. The order by which players are chosen is deterministic. The payoff to each player is a measurable function of the infinite history of the players' actions.

As mentioned before, the existence of an ϵ-equilibrium in this case was proven by Mertens (1987). However an existence of a subgame perfect equilibrium is still an open problem.

The goal of the research is to verify if the game has a subgame perfect equilibrium and under which assumptions. Further, we intend to study the case where the order by which players are chosen is non-deterministic.

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