COMPLEXITY: Exercise No. 1
due in two weeks

1. Prove or disprove:

   (a) $(2n)! = O(n!^2)$.

   (b) $f(n) = O(n) \Rightarrow 2^{f(n)} = O(2^n)$.

   (c) $\log(n!) = \Theta(n \log n)$.

   (d) If $f = o(g)$ then there is h such that $f = o(h)$ and $h = o(g)$.

   (e) If $f_1, f_2, \ldots, f_k \in O(n)$ then $\sum_{i=1}^{k} f_i(n) = O(n)$.

   (f) If $f_1, f_2, \ldots \in O(n)$ then $\sum_{i=1}^{n} f_i(n) = O(n^2)$.

   (g) There is a function f such that $f(n) = O(n^{1+\epsilon})$ for every $\epsilon > 0$ but $f(n) = \omega(n)$.

   (h) If f is monotone positive function,$f(n) = O(n)$ and $f(n) \neq o(n)$ then $f(n) = \Theta(n)$.

2. Let M be a k-tape DTM with input and output which always stops with time complexity $t(n)$ and space complexity $s(n)$. Prove that there is a constant c (which depends on M) such that $t(n) \leq nc^{s(n)}$ (assume $s(n) > 0$ for all n).

3. Describe explicitly an $O(\log k)$-space and $O(k)$-time 3-tape DTM that given a number k in binary encoding, it writes 1^k (i.e. k ones) on the output tape.

4. Sketch a DTM that accepts L_{pal} (the language of palindromes) with space $O(\log n)$ and time $O(n^2/\log n)$.

5. Prove that any 1-tape DTM with an infinite tape (i.e., the head of the machine can go any distance to the left or the right on the tape without limitations. Initially, the head is positioned on the first letter to the left of the input, and all the other letters on the tape are $\sqcup$) whose time complexity is $t(n)$ can be simulated by an ordinary 1-tape DTM with time complexity $t(n)$ (exactly!)

6. Show that a DTM with a two-dimensional "tape" (the tape is infinite in all four directions. Assume that in each step, the head can either stay in the same location or move one step left,right,up or down) whose time complexity is $t(n)$ can be simulated by a regular 3-tape DTM in $O(t(n)^2)$ time.

7. Show that any RAM that runs in time $t(n)$ can be simulated by a RAM without indirect addressing in time $O(t^c(n))$ for some constant c. (assume that the input to the original RAM is encoded in a single input register).