Algorithmic Methods	**Fall Semester, 2016/17**

Final Exam: February 7, 2017

Lecturer: Prof. Yossi Azar

Solve **4 out of the 5** questions. Write short but full and accurate answers. Each question should start on a new page and each of its parts should not exceed a page. No extra material is allowed.

1. We are given a graph $G = (V, E)$ with cost on the edges $c : E \to R^+$ and a parameter k. One need to choose a subset F of the edges of minimum total cost such that for each (non-trivial) cut there are at least k edges of F in the cut.
   (a) (15 points) Design a polynomial time algorithm for the relaxed version in which one can choose fractions of edges (and pay proportionally) and the total fractions on each cut is at least k.
   (b) (10 points) We also want that for cuts $(S, V - S)$ where $|S| \leq 3$ the total fractions of edges of F in the cut is exactly k (for all other cuts - it is still at least k). Design a polynomial time algorithm.
2. Consider an approximation algorithm for MAX-SAT which is based on LP and randomized rounding. Let x be the LP optimal fractional solution.
   (a) (18 points) Show that if we randomly round each variable independently according to the function $p_i = 0.24 + 0.52x_i$ then we get a 3/4-approximation for MAX-SAT. You can assume that for any integer $k \geq 1$ we have $(0.76 - 0.52/k)^k \leq 1/4$.
   (b) (7 points) Show how to de-randomize this algorithm into a deterministic 3/4-approximation algorithm.
3. Assume we are given a rooted tree where vertices may have arbitrary degrees. Each vertex has a unique label between 1 and n. Each vertex has to choose a color based only on local information such that the resulting coloring is a valid one. Each vertex has the knowledge of which out of its incident edges is directed toward the root (the root knows it is the root).
   (a) (13 points) Claim an upper bound on the number of rounds required to color the tree in 6 colors.
   (b) (7 points) Given a 6 vertex-coloring of the tree transform it to a 3-coloring in constant number of rounds. Hint: transform the 6-coloring into a new 6-coloring such that each vertex has neighbors of only 2 colors.
   (c) (5 points) Assume that the input tree is already colored properly with $\log^* n$ colors - claim an upper bound on the number of rounds for coloring the rooted tree in 3 colors.
4. We are given a graph $G = (V, E)$ with cost on the vertices $c : V \to R^+$ and penalty on the edges $p : E \to R^+$. A partial vertex cover is any subset $S \subseteq V$ and its cost is $\sum_{v \in S} c(v) + \sum_{e \in F} p(e)$ where F is the set of edges (u, v) for which neither $u \in S$ nor $v \in S$.
   (a) (12 points) Form an integer LP, relaxed it and round it deterministically to get 3-approximation polynomial time algorithm.
   (b) (13 points) Use local ratio to get 2 approximation polynomial time algorithm (do not forget to characterize the instances for each the optimal value equals 0).
5. We are given a graph $G = (V, E)$ where $|E| = m$ and set of k request (s_i, t_i) of unit demand for $i = 1, \ldots k$. One need to choose a path P_i from s_i to t_i for each i. The load of edge e is $L_e = |\{i | e \in P_i\}|$ and the goal is to minimize $\max_e L_e$.
   (a) (10 points) Form an integer LP for the problem and relaxed it to be solved in polynomial time.
   (b) (15 points) Round the solution (in polynomial time) to an integer solution and get $1 + \epsilon$ approximation assuming the value of OPT is at least $6 \log m / \epsilon^2$.
   Recall the Chernoff bound: let $X_1, \ldots X_n$ be random variables $X_i \in [0, 1]$ for all i, $E(X_i) = \mu_i$, $X = \sum_i X_i$ and $E(X) = \sum_i \mu_i \leq \mu$. Then $Pr(x > (1 + \delta)\mu) \leq e^{-\delta^2 \mu / 3}$ for $\delta \leq 1$.

The duration of the exam is 3 hours and 30 minutes. GOOD LUCK