Algorithmic Methods	**Fall Semester, 2018/19**

Final Exam: January 21, 2019

Lecturer: Prof. Yossi Azar

Solve **4 out of the 5** questions. Write short but full and accurate answers. Each question should start on a new page and each of its parts should not exceed a page. No extra material is allowed.

1. We are given n agents, agent i with demand a_i. We need to assign non-negatives weights to groups of size $\lfloor n/3 \rfloor$ agents such that the total weight of all groups that contain each agent is at least its demand. The goal is to minimize the sum of the weights over all the groups.
   (a) (7 points) Show that there exists an optimal solution in which at most n groups get positive weights.
   (b) (10 points) Show how to find the value of the optimal solution in polynomial time (no need to find the groups but only the optimal value).
   (c) (8 points) Now assume that we are only allowed to put positive weights on groups of size $\lfloor n/3 \rfloor$ that do not contain both agent $2i$ and agent $2i-1$ for any $1 \le i \le n/2$ (assume n is even). Find the new value of the optimal solution in polynomial time.

2. Consider an approximation algorithm for MAX-SAT which is based on LP and randomized rounding. Let x be the LP optimal fractional solution.
   (a) (18 points) Let $0 \le \alpha \le 0.1$. Show that if we randomly round each variable independently according to the function $p_i = (0.25 - \alpha) + (0.5 + 2\alpha)x_i$ then we get a 3/4-approximation for MAX-SAT. You can assume that for any integer $k \ge 1$ we have $((0.75 + \alpha) - (0.5 + 2\alpha)/k)^k \le 1/4$ for $0 \le \alpha \le 0.1$.
   (b) (7 points) De-randomize this algorithm into a deterministic 3/4-approximation algorithm.

3. (25 points) We are given a set of tasks where task i has a width b_i and a value v_i for $i \in \{1, 2, \ldots, n\}$. For some fixed k task i is associated with intervals set, $\{(x_i^1, y_i^1), (x_i^2, y_i^2), \ldots, (x_i^k, y_i^k)\}$ where $x_i^j < y_i^j$ for all $1 \le j \le k$. A feasible solution is a set $S \subseteq \{1, 2, \ldots, n\}$ and $j_i \in \{1, \ldots, k\}$ for each $i \in S$ such that for any t we have $\sum_{i \in S, x_i^{j_i} < t < y_i^{j_i}} b_i \le 1$. The total value of a solution is $\sum_{i \in S} v_i$. The goal is to find a feasible subset with maximum total value. Design a 5 approximation algorithm.

4. We are given n jobs and m unrelated machines. The load of job i on machine j is w_{ij} and its value is v_i. The load of a machine is the sum of the loads of the jobs assigned to it. For a given load T the goal is to assign subset of jobs of maximum total value such that the load on any machine is at most T.
   (a) (6 points) Write the appropriate Integer LP formulation and relax it.
   (b) (13 points) Round the LP to a solution with an optimal value with maximum load of at most $2T$ compared with OPT which has maximum load of at most T. NOTE: fractional jobs can be leaves.
   (c) (6 points) Get a 1/2 approximation for the problem while having a maximum load of at most T.

5. Let $G = (V, E)$ be a graph where $|E| = m$ and the capacity of edge e is $c(e) \ge \frac{10 \log m}{\epsilon^2}$. We are given $2k$ requests: for $1 \le i \le 2k$ request i of value $v_i > 0$ is to route a unit demand from vertex s_i to vertex t_i on a single path. For each $1 \le i \le k$ we may route AT MOST one request among requests $2i$ and $2i-1$ while maintaining capacity constraints. The goal is to route requests of maximum total value.
   (a) (8 points) Form an integer LP for the problem and relaxed it to be solved in polynomial time.
   (b) (17 points) Round the solution (in poly time) to an integer solution and get $1 + \epsilon$ approximation. Recall the Chernoff bound: let $X_1, \ldots X_n$ be random variables $X_i \in [0, 1]$ for all i, $E(X_i) = \mu_i$, $X = \sum_i X_i$ and $E(X) = \sum_i \mu_i \le \mu$. Then $Pr(x > (1 + \delta)\mu) \le e^{-\delta^2 \mu/3}$ for $\delta \le 1$.

The duration of the exam is 3 hours. GOOD LUCK