Algorithmic Methods **Spring Semester, 2020/21**

Final Exam: Jul 1, 2021

Lecturer: Prof. Yossi Azar

Solve **4 out of the 5** questions. Write short but full and accurate answers. Each question should start on a new page and each of its parts should not exceed a page. No extra material is allowed.

1. We are given an undirected connected graph $G = (V, E)$ with a non-negative capacity w_e on each edge $e \in E$. We are given $k \leq \binom{n}{2}$ pairs (s_i, t_i) and demand d_i. The cost of decreasing the capacity of an edge e by each unit is $c_e \geq 0$ for each $e \in E$. One need to decrease the capacities of the edges such that for any i the maximum flow between s_i and t_i is at most d_i. Assume that all $e \in E$ all c_e and w_e and d_i for all i are non negative integers (bounded by 2^n where $|V| = n$).
   (a) (18 points) Assume that the goal function to minimize the total cost over all edges. Form an LP for the problem and show how to solve it by a polynomial time algorithm (in n).
   (b) (7 points) Now the goal function is to minimize the maximum cost of decreasing any edge (we are allowed to decrease capacities ONLY by integer numbers). Show a poly-time algorithm without LP.
2. Consider an approximation algorithm for MAX-SAT which is based on LP and randomized rounding. Let x be the LP optimal fractional solution.
   (a) (18 points) Show that if we randomly round each variable independently according to the function $p_i = 1 - 4^{-x_i}$ for $x_i \leq 0.5$ and $p_i = 4^{x_i - 1}$ for $x_i \geq 0.5$ then we get an 3/4-approximation for MAX-SAT (use the fact that $1 - 4^{a-1} \leq 4^{-a}$ for any a).
   (b) (7 points) Show that this algorithm does not achieve a better than 3/4 approximation by providing for every n an example of a formula on n variables where each variable appears at least once and the algorithm achieves exactly 3/4 (using non-vertex optimal solution of the LP - partial points).
3. We are given n jobs and m unrelated machines. The load of job i on machine j is w_{ij}. The load of a machine is the sum of the weights of the jobs assigned to it. Each job i should be either assigned to some machine or we encounter a penalty of p_i for not assigning this job. The goal is to minimize the maximum load PLUS the total penalty of all unassigned jobs.
   (a) (7 points) Form an integer LP for the problem and relaxed it to be solved in polynomial time.
   (b) (18 points) Round the solution in poly time to design a 3 approximation algorithm for the problem. Remark: Better bound is possible. Worst bound gets partial points.
4. Let $G = (V, E)$ be an undirected graph where $|E| = m$ and the capacity of edge e is $c(e) \geq \frac{10 \log m}{\epsilon^2}$ for some fixed $\epsilon > 0$. We are given m subgraphs where subgraph H_i has value v_i. The goal is to find a subset of the subgraphs of maximum total value such that each edge is used by at most $c(e)$ subgraphs.
   (a) (7 points) Form an integer LP for the problem and relaxed it to be solved in polynomial time.
   (b) (18 points) Round the solution (in poly time) to an integer solution and get $1 + \epsilon$ approximation. Recall the Chernoff bound: let $X_1, \ldots X_n$ be random variables $X_i \in [0, 1]$ for all i, $E(X_i) = \mu_i$, $X = \sum_i X_i$ and $E(X) = \sum_i \mu_i \leq \mu$. Then $Pr(x > (1 + \delta)\mu) \leq e^{-\delta^2 \mu / 3}$ for $\delta \leq 1$.
5. We are given set of intervals, interval i which is (s_i, t_i) has a width $w_i \leq 1/2$ and a value v_i. The goal is to find subset of intervals such that the total width at any point is at most 1 with total maximum value. By using local ratio with the interval that ENDS FIRST (with width w and value $v > 0$) and using values of $\frac{v w_i}{1 - w}$ we get 2-approximation algorithms. Instead of taking the first interval assume we take an arbitrary (positive value) interval which does NOT contain any other interval.
   (a) (18 points) Define the exact algorithm and show that we get in this way a 4-approximation.
   (b) (7 points) Show by example that if we choose an interval that possibly contains other intervals the approximation ratio may be arbitrary large.

The duration of the exam is 3 hours. GOOD LUCK