Definition: A leaf $\gamma$ in a lamination $L$ is isolated if for each $x \in \gamma$ there exists a neighborhood $U$ of $x$ such that $(U, U \cap L)$ is homeomorphic to (disk, diameter).

Remark. If this condition holds for one $x \in \gamma$, it holds for all $y \in \gamma$. Indeed, if $\gamma$ is not isolated near $x \in \gamma$, then we have a sequence of geodesics $\gamma_n$ in $L$, with $x_n \in \gamma_n$ such that $x_n \to x$. Looking at their lifts to $\H^2$ (lift $x$ to some point $\tilde{x}$, and $x_n$ to $\tilde{x}_n$ so that $\tilde{x}_n \to \tilde{x}$), because $\gamma_n$ and $\gamma$ never intersect, the endpoints of the lifts $\tilde{\gamma}_n$ approach the endpoints of the lift $\tilde{\gamma}$, and we have that the geodesic $\gamma$ is not isolated near any point (this is essentially the continuity of directions given by lemma 3.1).

Definition: Set $L' := L - \{\text{isolated leaves}\}$. Note that $L'$ is closed. If $L'$ is non-empty, call it the derived lamination of $L$.

Lemma 3.6: If $L'$ is empty, $L$ is a finite union of simple closed geodesics, and $L$ is an isolated point in $\Lambda(F)$

Proof. If $L'$ is empty, all leaves of $L$ are isolated, so $L$ is a closed 1-submanifold (since isolated leaves must be closed). Each one is closed, so has an $\varepsilon$- neighborhood homeomorphic to an annulus. $L$ must have a finite number of leaves, otherwise the leaves have a limit point.

So, for $\varepsilon$ small enough, $\overline{N_\varepsilon(L)}$ is a disjoint union of annuli. Suppose $L_*$ is some lamination with $d(L, L_*) < \varepsilon$. Any leaf $\gamma$ of $L_*$ is thus contained in $N_\varepsilon(L)$, and therefore in $N_\varepsilon(C)$ for $C$ a closed leaf in $L$. Thus, some lift $\tilde{\gamma}$ is in $N_\varepsilon(\tilde{C}) \subseteq \H^2$. But the only geodesic in $N_\varepsilon(\tilde{C})$ is $\tilde{C}$ itself, so $\gamma = C$, and $L_* = L$ (which shows isolation).

Remark. We will see later on that there exists perfect laminations, $L' = L$, however coming up with one such example explicitly is quite hard. One such
example might be a lamination without any closed leaves (such as the one we’ve seen in the beginning of Avi’s lecture).

**Lemma 3.7:** Let $h : F_1 \to F_2$ be a homeomorphism of closed orientable hyperbolic surface, and let $\tilde{h} : \mathbb{H}^2 \to \mathbb{H}^2$ be a lift of $h$ to the universal cover. Then $\tilde{h}$ has a unique continuous extension to $\mathbb{H}^2 \cup S^1_\infty$.

**Remark.** The lift $\tilde{h}$ exists by the general lifting criterion: $(p_1 \circ h)_* \pi_1 \mathbb{H}^2 = 0 \subseteq (p_2)_* \mathbb{H}^2 = 0$.

**Proof.** The idea of the proof is to take $\gamma$, a geodesic in $\mathbb{H}^2$, and show that as one approaches the ends of $\gamma$ in $S^1_\infty$, $\tilde{h}(\gamma)$ converges to a point in $S^1_\infty$.

Without loss of generality, suppose $\tilde{h}(O) = O$ (otherwise, compose with appropriate isometries to move everything to the origin, and since isometries can be extended to $S^1_\infty$, you can compose with their inverses after the extension of $\tilde{h}$).

Notice that $\tilde{h}$, $\tilde{h}^{-1}$ are lifts of continuous maps on compact surfaces, thus they are uniformly continuous with respect to the hyperbolic metric. To see this, take a (compact) fundamental domain of $F_1$ in $\mathbb{H}^2$, and take a closed neighborhood of this domain. Then the function $\tilde{h}$ is uniformly continuous in this neighborhood. Now, for any two (close) elements in $\mathbb{H}^2$, just move them so one is in the fundamental domain, and the other is in the neighborhood, and use uniform continuity established in that closed neighborhood.

Hence, there exists a $k > 0$ such that:

\[
d(x, y) \leq 1/k \implies d(\tilde{h}x, \tilde{h}y) \leq 1
\]

\[
d(\tilde{h}x, \tilde{h}y) < 1/k \implies d(x, y) < 1
\]

By subdividing the geodesic section $x$ and $y$ into $k$ equal subintervals, it follows:

\[
d(x, y) \leq 1 \implies d(\tilde{h}x, \tilde{h}y) \leq k
\]

\[
d(\tilde{h}x, \tilde{h}y) < n \implies d(x, y) < kn
\]

Let $x \in S^1_\infty$, and let $\gamma$ be the geodesic from $O$ to $x$. Let $P_t = P_t(x)$ be a point on $\gamma$ a distance $t$ from $0$, and $Q_t = \tilde{h}(P_t)$. Since $d(P_t, 0) \geq t$, we know that $d(Q_t, O) \geq t/k$, and in particular approaches $\infty$ as $t \to \infty$.  

2
Pick a reference line through $O$ and let $\theta_t$ be the angle between $OQ_t$ and the reference line. We want to show that $\theta_t$ converges. Then it will follow that $Q_t$ converges to a point in $S^1_\infty$, and we will define $hx = \lim_{t \to \infty} Q_t$.

By the second inequality, $Q_t$ and $Q_{t+1}$ have distance $\geq t/k$ from the origin. By the first inequality, $d(Q_t, Q_{t+1}) \leq k$. Hence, by the triangle inequality, every point on the geodesic arc between $Q_t$ and $Q_{t+1}$ has distance $\geq t/k - k$ from $O$. When $t \geq 2k^2$, the distance is at least $t/2k$.

Let $R_t, R_{t+1}$ be the points on the circle of radius $t/2k$ with angle $\theta_t, \theta_{t+1}$ from the reference line, resp. (so $Q_t R_t$ and $Q_{t+1} R_{t+1}$ are straight lines through the origin). The projection from the geodesic arc $Q_t Q_{t+1}$ to the geodesic arc $R_t R_{t+1}$ decreases length (every point on $Q_t Q_{t+1}$ has radius $\geq t/2k$ so contributes to a length element greater than the length element on the circular (non-geodesic) arc $R_t R_{t+1}$), so:

$$k \geq d(Q_t, Q_{t+1}) \geq \arcsin(\frac{\theta_{t+1} - \theta_t}{\sinh(t/2)})$$

Where $2\pi \sinh(t/2k)$ is the circumference of the circle with radius $t/2k$, a calculation we’ve seen in chapter 1. Since $\sinh x \geq e^x/4$ when $x \geq 1$, we get:

$$|\theta_{t+1} - \theta_t| \leq 4ke^{-t/2k}$$

For $u \geq t \geq 2k^2$ (by using integral-sum inequalities):

$$|\theta_u - \theta_t| \leq \int_{t-1}^{[u]+1} 4ke^{-s/2} ds \leq Ce^{-t/2k}$$

Where $C$ is a constant (only depends on $k$). Cauchy’s criterion shows that $\theta_t$ converges, and so we are done.

Why is this continuous? A neighborhood basis of $\tilde{h}(x)$ is the collection of $2\varepsilon$ sections of the exterior of a circle with hyperbolic radii $\rho$. Take one of these neighborhoods $U$. 
By choosing $t_0$ large enough ($t_0 > 2k^2$ and $t_0 > \rho k$) we can ensure the exterior of the circle with radius $t_0$ is sent to the exterior of the circle with radius $\rho$.

Also choose $t_0$ so that $Ce^{-t_0/2k} < \varepsilon/3$. In particular, points on the same diameter (with radius $> t_0$) will be mapped by $\tilde{h}$ to points that have an angle difference of less than $\varepsilon/3$ (follows from $|\theta_u - \theta_t| \leq Ce^{-t/2k}$).

Let $A = P_{t_0}$. $\tilde{h}A$ differs from $\tilde{h}x$ in angle at most $\varepsilon/3$. By continuity of $\tilde{h}$, for $\delta$ small enough, the arc of length $2\delta$ centered at $A$ of the circle with radius $t_0$ gets sent to points $\varepsilon/3$ close to $\tilde{h}(A)$ (in angle). Now, take any point on the section of the exterior of the circle with radius $t_0$. Its image differs by less than $3 \cdot \varepsilon/3 = \varepsilon$ radians from the point $\tilde{h}x$, so it is in $U$.

The uniqueness follows immediately from the fact that $\tilde{h}(\gamma)$ converges to a unique point. \hfill \Box

**Lemma 3.8:** If $h_0, h_1 : F_1 \to F_2$ are homotopic homeomorphisms between compact orientable hyperbolic surfaces, and $\tilde{h}_0$ is a lift of $h_0$, then there is a lift $\tilde{h}_1$ of $h_1$ such that $\tilde{h}_0 = \tilde{h}_1$ on $S^1_{\infty}$.

**Remark.** The lift $\tilde{h}$ of $h$ to $H^2$ is not unique. It depends on choice of basepoint. However, once $\tilde{h}$ has been chosen, its extension to $S^1_{\infty}$ is unique. The lemma says that you can choose $\tilde{h}_1$ in this way.

**Proof.** Let $H : F_1 \times I \to F_2$ be a homotopy between $h_0, h_1$. Let $\hat{H}$ be the lift of $H$ so that $\hat{H}_0 = \hat{h}_0$. $H$ is uniformly continuous, hence $\hat{H}$ is also uniformly continuous (follows from the same periodicity argument in lemme 3.7) so the hyperbolic length of $\hat{H}(a \times I)$ are bounded for $a \in F_1$. The hyperbolic distance between $h_0(a) = H(a, 0)$ and $h_1(a) = H(a, 1)$ is bounded by the length of
\( \tilde{H}(a \times I) \), hence is bounded for any \( a \). So the euclidean distance, as \( a \) tends to \( S^1_\infty \), this means the euclidean distance between \( \tilde{h}_0(a) \) and \( \tilde{h}_1(a) \) tends to 0, so they agree on the boundary \( S^1_\infty \).

\[ \]

**Question:** Does \( \text{Aut}(F) = \pi_0(\text{Homeo}^+(F)) \) act “naturally” on \( F \)?

More precisely, we have a map \( \Phi : \text{Homeo}^+(F) \to \text{Aut}F \) projecting every homeomorphism to the set of homeomorphisms isotopic to it. We ask if there is a homomorphism \( \Psi : \text{Aut}(F) \to \text{Homeo}^+(F) \) such that \( \Phi \circ \Psi = 1_{\text{Aut}(F)} \)? This was an open question for many decades:

1. First, it was shown that if \( \pi \subseteq \text{Aut}(F) \) is finite, there exists a \( \Psi : \pi \to \text{Homeo}^+(F) \) such that \( \Phi \circ \Psi = 1_\pi \). This was proved by Nielsen and Fenchel in the case that \( \pi \) is cyclic, and more generally solvable. Kerchoff further proved this for all finite \( \pi \) using a Thurston’s compactification of Teichmüller spaces.

2. Morita showed there is no \( \Psi : \text{Aut}(F) \to \text{Diffeo}(F) \) satisfying the condition.

3. Gromov and Cheeger proved that \( \text{Aut}F \) does act naturally on the unit tangent bundle \( UT(F) \) which is defined like \( PT(F) \), including the orientation of the geodesic. We will see a version of this soon.

4. In 2007 Markovic showed the answer is negative for genera \( g > 5 \). The question “(likely) remains open for genera 2, 3, 4 and 5.

**Lemma 3.9:** If \( F = \mathbb{H}^2 / \Gamma \) is a compact orientable surface, and \( UT(F) = \{ (x, \sigma) : \sigma \) is an oriented geodesic about \( x \) with length 2 \}, then \( UT(F) = Y / \Gamma \) where

\[
Y = \{ (a, b, c) : a, b, c \) are distinct points in a counter-clockwise order on \( S^1_\infty \}
\]

**Proof.** Given \( (a, b, c) \in Y \), let \( x \) be the foot of the perpendicular from \( c \) to the geodesic \( ab \), and let \( \sigma \) be the length 2 segment of \( ab \) centered on \( x \). Set \( q(a, b, c) = (p(x), p(\sigma)) \)
This map is clearly continuous, and for \( g \in \Gamma \):

\[ q(g(a, b, c)) = q(ga, gb, gc) = (p(gx), p(g\sigma)) = (p(x), p(\sigma)) = q(a, b, c) \]

Where \( ga, gb, gc \) map to \( gx, g\sigma \) because \( g \) perserves all geodesics and angles. This means \( q \) induces a well defined map \( Y/\Gamma \to F \), and it is a homeomorphism because given \( (x, \sigma) \), there is a unique geodesic in \( \mathbb{H}^2 \) corrsponding to \( \sigma \), upto action of \( \Gamma \).

**Theorem 3.10:** Any orientation perserving homeomorphism \( h : F_1 \to F_2 \) of closed hyperbolic surfaces induces a homeomorphism \( \hat{h} : UT(F_1) \to UT(F_2) \).

If \( h, k \) are homotopic, then \( \hat{h} = \hat{k} \). If \( h_1 : F_1 \to F_2 \) and \( h_2 : F_2 \to F_3 \) are orientation perserving homeomorphisms, then \( \hat{h_2} \circ \hat{h_1} = \hat{h_2} \circ \hat{h_1} \). Finally, \( \hat{h} \) carries lifted oriented geodesics to lifted oriented geodesics.

**Proof.** Let \( F_1 = \mathbb{H}^2/\Gamma_1 \), and let \( \hat{h} : \mathbb{H}^2 \to \mathbb{H}^2 \) be a lift of \( h \). By a previous theorem, \( \hat{h} \) has a unique extension to \( S^1_\infty \). Let \( \hat{h} : Y \to Y \) be the map induced by \( \hat{h} \).

If \( g_1 \in \Gamma_1 \) then \( \hat{h}g_1 = g_2\hat{h} \) where \( g_2 = \hat{h}g_1\hat{h}^{-1} \in \Gamma_2 \) (this follows from the commutativity of the diagram of the lift \( \hat{h} \)). It follows that \( \hat{h} \) induces a map \( \hat{h} : Y/\Gamma_1 \to Y/\Gamma_2 \), which is independent of choice of lift of \( h \) (any other lift \( \hat{h} \) satisfied \( \hat{h} = g_2\hat{h} \) for \( g_2 \in \Gamma_2 \)). By applying the previous lemma, we obtain a map \( \hat{h} : UT(F_1) \to UT(F_2) \).

If \( h, k \) are homotopic, there are lifts of both which agree on the boundary, thus they induce the same map. Moreover, we can lift \( h_2 \circ h_1 \) to \( \hat{h}_2 \circ \hat{h}_1 \), thus having \( \hat{h}_2 \circ \hat{h}_1 = \hat{h}_2 \circ \hat{h}_1 \).

Now, let \( \gamma \) be a lifted geodesic in \( UT(F_1) \) (with orientation chosen arbitrarily). I.e., each point \( (x, \sigma) \in \gamma \) satisfies \( x \) is in the original geodesic, and \( \sigma \) is in the direction of the geodesic. Applying the homeomorphism \( UT(F_1) \to Y/\Gamma_1 \), we can see that the geodesic \( \gamma \) is sent to the set of points \( \{(a, b, c) : c \in \text{arc}(a, b)\}/\Gamma_1 \) where \( a, b \) are endpointsw of some lift of \( \gamma \) in \( \mathbb{H}^2 \). \( \hat{h} \) maps this set to

\( \{(\hat{h}a, \hat{h}b, \hat{h}c) : c \in \text{arc}(a, b)\}/\Gamma_2 \)

Since \( \hat{h} \) perserves orientation and is continuous, so \( \hat{h} \) (\( \text{arc}(a, b) \)) = \( \text{arc}(\hat{h}a, \hat{h}b) \) , so the image is equal to

\( \{(\hat{h}a, \hat{h}b, c) : c \in \text{arc}(\hat{h}a, \hat{h}b)\}/\Gamma_2 \)

which is exactly a lifted geodesic in \( UT(F_2) \) (the one with endpoints \( \hat{h}a, \hat{h}b \)). \( \square \)

**Remark.** If \( \gamma \) is any geodesic in \( F_1 \), it can either be oriented \( \gamma_+ \) or \( \gamma_- \). \( \hat{h}(\gamma_+) \) and \( \hat{h}(\gamma_-) \) are both sent to the same (unoriented) geodesic: if (some lift of) \( \gamma \)
has endpoints \( P, Q \) in \( \mathbb{H}^2 \), then (some lifts of) \( \hat{h}(\gamma_+) \), \( \hat{h}(\gamma_-) \) will have endpoints \( \tilde{h}P, \tilde{h}Q \), showing they map to the same geodesic:

\[ \begin{align*}
\text{Theorem 3.11:} & \quad \text{Any orientation preserving homeomorphism } h : F_1 \to F_2 \\
& \text{of closed oriented hyperbolic surfaces induces a homeomorphism } \hat{h} : \Lambda(F_1) \to \Lambda(F_2). \text{ If } h \text{ is homotopic to } k, \text{ then } \hat{h} = \hat{k}. \text{ If } h_1 : F_1 \to F_2 \text{ and } h_2 : F_2 \to F_3 \text{ are orientation preserving homeomorphisms, then } (h_2 \circ h_1) = \hat{h}_2 \circ \hat{h}_1.
\end{align*} \]

Proof. Given an unoriented geodesic \( \gamma \) in \( F_i \), we can define its lift to \( UT(F_i) \) as:

\[ \hat{\gamma} = \{(x, \sigma) : x \in \gamma, \sigma \text{ is a segment of } \gamma \text{ centered at } x \text{ oriented in one of the two possible orientations}\} \]

(So \( \hat{\gamma} \) is a two-covering of \( \gamma \)), and let \( q_i : UT(F_i) \to F_i \) be the natural projection.

Define \( \hat{h}(L) = \bigcup_{\gamma \subseteq L} q_2 \left( \hat{h}(\hat{\gamma}) \right) \). By the previous remark, \( \hat{h}(\hat{\gamma}) \) is a lifted geodesic in \( UT(F_2) \). Let \( \hat{\gamma}_1, \hat{\gamma}_2 \) be lifted leaves of \( L \) with endpoints \( a_1, b_1 \) and \( a_2, b_2 \). Since \( \hat{\gamma}_1, \hat{\gamma}_2 \) do not intersect, \( a_1, b_1 \) do not separate \( a_2, b_2 \). \( h \) perserves orientation, so \( \tilde{h}(a_1), \tilde{h}(b_1) \) do not separate \( \tilde{h}(a_2), \tilde{h}(b_2) \). Hence, \( \hat{h}(L) \) is composed of simple leaves which do not intersect.

By theorem 3.10, \( \hat{h} \) is continuous, and \( q_2 \) is also continuous. Since \( \hat{L} = \bigcup_{\gamma \subseteq \hat{\gamma}} \hat{\gamma} \) is compact (a 2-covering of \( L \) which is compact), it follows that \( \hat{h}(L) \) is compact, and hence a lamination.

Finally, \( h \) is a continuous, since \( h = \left( \left( q_2 \circ \hat{h} \right) \right) \left( q_1 |_{\Lambda(F_1)} \right)^{-1} \), where \( \hat{\Lambda}(F) \) is the set of lifted laminations in \( F \). Define \( \left( h^{-1} \right) \) similarly, and see that they are inverses, and that shows \( \hat{h} \) is a homeomorphism.

The other two properties follow directly from the fact \( \hat{h} \) also satisfies those properties. \( \square \)
**Exercise:** Let $F = \mathbb{H}^2/\Gamma$ be a hyperbolic surface, and $Y$ as defined in lemma 3.9. We can define an equivalence relation on $Y$, setting $(a,b,c) \sim (b,a,d)$ when the geodesics $ab$ and $cd$ are perpendicular. Let $Z = Y/\sim$ be the quotient space. Given $[(a,b,c)] \in Z$, let $x$ be the foot of the geodesic from $c$ perpendicular to $ab$, and let $q : Z/\Gamma \to PT(F)$ send $[(a,b,c)]$ to $(p(x), p(\sigma))$ where $\sigma$ is the (unoriented) segment of $ab$ centered at $x$ with length 2, and $p : \mathbb{H}^2 \to F$ is the covering map. See Avi’s summary for the definition of $PT(F)$. Look again at lemma 3.9, this is the same as the construction in that lemma, with everything unoriented.

1. Show $q$ is a well defined homeomorphism.

2. Let $h : F_1 \to F_2$ be an orientation preserving homeomorphism, and $\tilde{h} : \mathbb{H}^2 \cup S^1_{\infty} \to \mathbb{H}^2 \cup S^1_{\infty}$ be a lift and its extension (see lemma 3.7). Show that $\tilde{h}$ does not necessarily induce a homeomorphism on $Z/\Gamma_1$, i.e., there does not exist an $\hat{h} : Z/\Gamma_1 \to Z/\Gamma_2$ so that $\hat{h}$ sends $[(a,b,c)]$ to $[(\tilde{ha}, \tilde{hb}, \tilde{hc})]$. (Hint: It is enough to find a homeomorphism $\tilde{h}$ which does not send identified points in $Z$ to the same image, don’t worry too much about the original homeomorphism $h$).

Remark. This exercise shows why we’ve chosen $UT(F)$ in theorem 3.10.