Definition Surface

A surface is a compact, connected, oriented 2-dimensional topological manifold. We call a surface closed if it has no boundary\(^1\).

Definition Hyperbolic Structure

Let \( F \) be a surface. A Hyperbolic Structure on \( F \) is an atlas of charts \( \phi_\alpha : U_\alpha \to \mathbb{H}^2 \) such that restriction of the transition map \( \phi_i \circ \phi_j^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j) \) is an orientation preserving isometry of \( \mathbb{H}^2 \).

We usually require \( F \) and the atlas to be oriented.

Notice that are defining a geometry on \( F \)!

This is because at every point \( x \in F \) we can choose a map \((U, \varphi)\) from the atlas, and a neighborhood \( U_x \subseteq U \) of \( x \), and pull back the Riemannian metric \( ds^2 \) using \( \varphi \). This is well defined since all of the intersections of different neighborhoods containing \( x \) are isometric, as by the definition of the atlas.

Example Every closed oriented surface \( F \) of genus \( \geq 1 \) has a hyperbolic structure

Let \( F \) be a surface of genus \( g > 1 \). \( F \) can be made by identifying edges of a \( 4g \)-gon (so that all of the vertices are identified):

![Figure 1: Construction of surfaces from 4g-gons [6]](image)

And let \( P \) be a regular hyperbolic \( 4g \)-gon with internal angles \( \frac{\pi}{2g} \) in \( \mathbb{H}^2 \) concentric with the Poincaré disc model. To show that such a polygon exists, we can consider the hyperbolic polygon \( P_\ell \), whose vertices are the points of intersection between

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\(^1\)For a manifold with a boundary, we allow the boundary points to be homeomorphic to a "half" ball.
$4g$ equispaced geodesic rays from the origin $O$, and the hyperbolic circle of center $O$ with hyperbolic radius $t > 0$. As $t$ increases, the internal angle of $P_t$ decreases from $(4g - 2)\pi$ (the Euclidean value, here it is necessary that $g > 1$), down to $0$. By continuity, we get the desired outcome.

As the sides of $P$ have the same length, we can glue them together with orientation preserving isometries. To give $F$ a hyperbolic atlas, we notice that the internal points of the polygon have an open neighborhood that remains an open neighborhood, even after the application of the isometries; so the “problematic” points are the ones on the boundary.

For points on the sides of the polygon that are not vertices, we use the unique isometries gluing the edges to make charts for the edges. (Here we would like to use a continuity argument to show that both of the “glued” sides have the same values.) And for the vertices one can show that because the angle sum is $4g \cdot \frac{\pi}{2g} = 2\pi$ we can get a chart for the vertices, this is left as an exercise.

For example, for a surface of genus 1 we can choose the charts:

$$
\tilde{\phi}_1(x, y) = (x, y)
$$
$$
\tilde{\phi}_2(x, y) = \begin{cases} (x, y + 1) & \text{if } y \leq 0 \\ (x, y - 1) & \text{if } y > 0 \end{cases}
$$
$$
\tilde{\phi}_3(x, y) = \begin{cases} (x + 1, y) & \text{if } x \leq 0 \\ (x - 1, y) & \text{if } x > 0 \end{cases}
$$
$$
\tilde{\phi}_3(x, y) = \begin{cases} (x + 1, y + 1) & \text{if } x \leq 0, y \leq 0 \\ (x + 1, y - 1) & \text{if } x \leq 0, y > 0 \\ (x - 1, y + 1) & \text{if } x > 0, y \leq 0 \\ (x - 1, y - 1) & \text{if } x > 0, y > 0. \end{cases}
$$

Figure 2: charts defining a structure on the flat surface of genus 1 [2]

Figure 3: charts defining a structure on the hyperbolic surface of genus 2 [2]
Definition  Complete² Hyperbolic Surface

A hyperbolic surface $F$ is complete, if it is complete as a metric space. Recall that we can define a metric on $F$ by pulling back the Riemannian metric on $\mathbb{H}^2$ via the maps in the atlas, and from the pulled back Riemannian metric, we can induce a metric that is defined as in a metric space. In this case we require that the induced metric be complete.

Remark

The isometries that identify each of the matching edges in the construction above, generate a group. The images of the $4g$-gon of all of the members of the group, form a tiling³ of the Poincaré disk model!

![Figure 4: Tiling of the hyperbolic plane in the Poincaré disc model [3]](image)

Geodesics in $F$ are defined locally. I.e. we require that every point on the chart have an open neighborhood where it is pulled back from a geodesic in $\mathbb{H}^2$ (via the map):

![Figure 5: Pulling back a geodesic [4]](image)

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²In Hebrew: שלם
³I.e. an edge-to-edge filling of the hyperbolic plane which has polygons as faces, where there is an isometry mapping any vertex onto any other.
This leads us to the following lemma:

**Lemma 2.1  The easy half of the Hopf-Rinow theorem**

In a complete hyperbolic surface, all geodesics can be extended indefinitely.

**Proof**

Assume by contradiction that we have a bounded arc, which is a geodesic, on a complete surface $F$. Take a sequence of points that converges to an end point $a$, this sequence is Cauchy, so from the completeness of $F$ we have that $a$ (which is an endpoint) is also in $F$. Now, take a chart that maps $a$ to the center of the Poincaré disk; in the disk this geodesic (or its image) can be extended, and we can pull this extension back to $F$ via the chart map, thus extending the map. This process can be done indefinitely, and will never converge since we can apply this process to the point we are converging to, This is in contradiction to the arc being bounded.

**Theorem 2.2**

Any complete, connected, simply-connected hyperbolic surface is isometric to $\mathbb{H}^2$.

**Proof**

Let $\{(U_i, \varphi_i)\}$ be an atlas describing a geometric structure on $F$. We will define the following isometries:

![Figure 6: E and D](image)

**The Exponential map** - $E : \mathbb{H}^2 \rightarrow F$

Choose $A \in U_0$ such that $\varphi_0(A) = O$ (The center of the Poincaré disk). If no such $A$ exists, we can apply an isometry that will move an arbitrary point to $O$, and change $\varphi_0$ to be the composition with this isometry. For each $X \in \mathbb{H}^2$, we can extend the geodesic $\varphi_0^{-1}(OX)$ to a complete geodesic (like we saw in the lemma!)

We define $E(X)$ as the point on the extended geodesic with $\text{dist}(A, E(X)) = \text{dist}(O, X)$.

**The Developing map** - $D : F \rightarrow \mathbb{H}^2$

We will inductively construct $D$ by an “analytic continuation” of $\varphi_0$:

Fix $p \in U_0$. For each $q \in F$ we choose a path $\gamma$ from $p$ to $q$ in $F$. Next, we cover $\gamma$ with a finite sequence of charts: $U_0, U_1, \ldots U_n$ with appropriate maps $\varphi_i : U_i \rightarrow \mathbb{H}^2$, such that $U_i \cap U_{i+1}$ is connected (from the atlas):
We now choose points $X_i \in \gamma$ s.t. $[x_i, x_{i+1}] \subseteq U_i$.

For our inductive assumption suppose that $\varphi_j \equiv \varphi_{j-1}$ in the component of $U_j \cap U_{j-1}$ containing $x_j$, for all $j = 1, \ldots, i$. (This is true for the case $j = 0$ immediately)

For the inductive step, assume that $\varphi_{i+i} \neq \varphi_i$ on the component of $U_i \cap U_{i+1}$ containing $x_i$. (O.w. we continue.) Because $\varphi_i$ and $\varphi_{i+1}$ are from an atlas describing a geometric structure, we get that $\varphi_{i+1} = g_i \circ \varphi_{i-1}$, for some unique isometry $g_i$ of $\mathbb{H}^2$. We replace $\varphi_{i+1}$ by $g_i^{-1} \circ \varphi_{i-1}$, and now the maps agree on the intersection.

Finally, set $D(q) = \varphi_n(q)$.

Next, it is shown in [2], that it is possible to prove that the value of $D(q)$ depends only on $(U_0, \varphi_0)$ and the homotopy class of $\gamma$. And that because $F$ is simply connected we get that $D$ is well defined.

Finally, we notice that $D \circ E = \text{id}_{\mathbb{H}^2}$. We show this by choosing the path $\gamma$ to be the geodesic, and since geodesics are defined locally, and are unique (locally, there is only one shortest path), so in our construction of $D$ we will get the same path as in the lemma. And therefore, the same final value of the function.

Next, notice that $\text{im}(E)$ is closed. But since $E \circ D$ is a retraction, $\text{im}(E)$ is open from invariance of domain\(^4\), and $F$ is connected, so $E \circ D = \text{id}_F$.

**Corollary**

This implies that the universal cover of a compact hyperbolic surface is $\mathbb{H}^2$.

Before we continue, we need some more background from algebraic topology:

**Definition** *Deck Translations*

Let $p : \tilde{X} \to X$ be a covering space. We define the *deck translations* as the group of homeomorphisms that preserve the projection $p : \{ \tau : \tilde{X} \to \tilde{X} \mid \tau \text{ is a homeomorphism, and } p \circ \tau \}$, under composition.

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\(^4\)This is the topological equivalent of the open mapping theorem:

Let $U \subseteq \mathbb{R}^n$ be an open set, and let $f : U \to \mathbb{R}^n$ be an injective continuous map, the $V := f(U)$ is open in $\mathbb{R}^n$, and $f$ is a homeomorphism between $U$ and $V$. 

5
Theorem

The deck translations are isomorphic to $\Pi_1(X, x_0)$, and therefore are isomorphic to a subgroup $\Gamma$ of $\text{PSL}_2(\mathbb{R})$.

Additionally, since every $\varepsilon$-neighborhood of $x_0$ has at most a discrete number of preimages in $\tilde{X}$, we get that $\Gamma$ is also discrete. One can also show that all of the elements of $\Gamma - \{1\}$ are hyperbolic, and that $\Pi_1(X)$ acts freely on $X$.

Therefore, $\Pi_1(X)$ cannot have $\mathbb{Z} \times \mathbb{Z}$ as a subgroup (since $\text{PSL}_2(\mathbb{R})$ does not have such a discreet subgroup), and in particular we cannot define a hyperbolic structure on the torus. (We can identify the torus with $S^1 \times S^1$, and it is possible to prove that the fundamental group of a product of spaces is the product of the fundamental groups, and that the fundamental group of $S^1$ is isomorphic to $\mathbb{Z}$.)

Definition  An Essential Closed Curve

A closed curve in a surface is essential if it is not null-homotopic (i.e. not homotopic to a constant map.)

Lemma 2.3

Every essential closed curve in a closed hyperbolic surface is freely\(^{5}\) homotopic to a unique closed geodesic.

Proof

Let $C$ be an essential curve in $F$. Choose an $x$ on $C$, and an $\tilde{x} \in \tilde{F} = \mathbb{H}^2$ such that $p(\tilde{x}) = x$. Let $\bar{C}$ be a curve in $\mathbb{H}^2$ containing $\tilde{x}$ and projecting to $C$ (i.e. $p(\bar{C}) = C$).

Since the curve is closed and not null homotopic, its lift in $\bar{C}$ contains (at least) two preimages of $x$, and each of these preimages define a $g \in \Gamma$. We can choose a preimage of $x$ and its matching $g$ such that projection of the segment $[\tilde{x}, g\tilde{x}]$ goes around $C$ once. Since $g$ is hyperbolic, meaning it has two fixed points in $S^\infty_1$, that define a geodesic axis $\tilde{\gamma}$. The image of $\tilde{\gamma}$ is a closed geodesic $\gamma$ in $F$. Choose any point $\tilde{y}$ on $\tilde{F}$ and any path $\tilde{U}$ connecting $\tilde{y}$ to $\tilde{x}$. The circuit from $\tilde{y}$ to $\tilde{x}$, along $\bar{C}$ to $g\tilde{x}$, along $g\tilde{U}$ to $g\tilde{y}$, and back to $\tilde{y}$ on the geodesic $\gamma$ bounds a singular “rectangle” in $\mathbb{H}^2$, which projects to a singular annulus\(^6\) in $F$. This annulus defines a free homotopy from $C$ to $\gamma$. 

\[\text{Diagram}\
\]
Now to prove the uniqueness, let $C$ be an essential closed curve freely homotopic to a closed geodesic $\gamma$, with the free homotopy $f: S^1 \times I \to F$. As before, let $\tilde{C}$ be a "component" of the preimage of $C$. Let $\tilde{f} : \mathbb{R} \times I \to \mathbb{H}^2$ be the lift of this map such that $f(\mathbb{R} \times \{0\}) = \tilde{C}$. Let $\tilde{\gamma} = \tilde{f}(\mathbb{R} \times \{1\})$ have endpoints $P, Q$. Now, since $S^1$ is compact, there exists a $d > 0$ that is an upper bound for the hyperbolic length of the arcs $f(z \times I)$ for $z \in S^1$. Therefore, in the Euclidian metric on the Poincaré disk, $\text{dist}(\tilde{C}, \tilde{\gamma}) \to 0$ as one approaches $P$ or $Q$. In particular, if $C$ and $\gamma$ are freely homotopic geodesics, the have the same endpoints and therefore coincide.

**Definition** closed 1-submanifold and essential 1-submanifold

A closed 1-submanifold of a surface $F$ is a disjoint union of simple closed curves in $F$. An essential 1-submanifold is a closed 1-submanifold in which every component is essential and no two components are homotopic.

Notice that since $F$ is compact, then the number of components is finite. (Otherwise we would have an accumulation point.)

**Lemma 2.4**

Every essential 1-submanifold $C$ of a closed hyperbolic surface is isotopic to a unique geodesic 1-submanifold.

**Proof**

Every component of $C$ is homotopic to a unique closed geodesic (by lemma 2.3). All of these geodesics are unique, since no 2 components of $C$ are homotopic; let us denote their union by $\gamma$. Notice that the preimage of $\gamma$ in $\mathbb{H}^2$ is obtained from the preimage of $C$ in $\mathbb{H}^2$, by replacing each component with the geodesic having the same endpoints in $S^1_\infty$ (as we saw in the proof of the uniqueness). Since no two components $\tilde{C}_1, \tilde{C}_1$ of the preimage of $C$ intersect, their endpoints do not separate each other, so the geodesics corresponding to them also do not intersect. Therefore, the preimage of $\gamma$ is a disjoint union of geodesics, and hence simple.

To prove uniqueness we will need the following definition and lemma:

**Definition** minimal intersection of essential 1-submanifolds

Essential 1-submanifolds $C_1, C_2$ have minimal intersection if they intersect transversely and if there do not exists any arcs $\gamma_1, \gamma_2$ in $C_1, C_2$ respectively, having common endpoints and such that $\gamma_1 \cup \gamma_2$ is the boundary of a disk in $F$.

**Lemma 2.5**

Let $C_1, C_2$ be essential 1-submanifolds of a closed surface, then $C_2$ is isotopic to an essential 1-submanifold having minimal intersection with $C_1$.

**Proof**

Since we are considering isotopy classes, we can assume that $C_1$ and $C_2$ are transverse (if not we can alter the curves slightly via an isotopy). If they have non-

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1In a free homotopy, we allow the endpoints to move "freely"
2In Hebrew: טבעת
3I.e. does not intersect itself
minimal intersection, therefore exists is an embedded disk $D_0$ in $F$ bounded by one subarc of $C_1$ and one of $C_2$. By the compactness of $F$ and transversality of the intersection, we get that the intersection $(C_1 \cup C_2) \cap D$ is a finite graph, if we think of the intersection points as vertices. Thus there is an innermost disk, that is, an embedded disk $D$ in $F$ bounded by one subarc of $C_1$ and one of $C_2$, with no other arcs passing through the interior of $D$:

![Figure 9: An innermost disk between two arcs [5]](image)

Now, via an isotopy, we can “push $C_2$ across $D$”, thus reducing the number of intersections of $C_1$ and $C_2$. We can continue this process until $C_1, C_2$ have minimal intersection.

Returning to the proof of lemma 2.4 -

According to the lemma we just proved, we can move $C$ via an isotopy, until $C$ and $\gamma$ have minimal intersection. Choose a component $\tilde{C}$ of the preimage of $C$ in $\mathbb{H}^2$. Since $C \cap \gamma$ has minimal intersection, $\tilde{C}$ intersect each component of the preimage of $\gamma$ at most once.

Note that $\tilde{C}$ has the same endpoints as a component $\tilde{\gamma}$ of the preimage of $\gamma$, and that there is a deck translation leaving $\tilde{C}$ and $\tilde{\gamma}$ invariant; this follows from the proof of lemma 2.3, where the unique geodesic is the one that the deck translation $g$ leaves invariant. Therefore, if $\tilde{C} \cap \tilde{\gamma}$ is not empty, it contains an orbit of point in the intersection under this deck translation. So therefore the intersection is infinite, in contradiction to the fact that $|\tilde{C} \cap \tilde{\gamma}| \leq 1$, so $\tilde{C} \cap \tilde{\gamma}$ is empty.

For any other component $\tilde{\gamma}'$ of the preimage of $\gamma$, the intersection $|\tilde{C} \cap \tilde{\gamma}'|$ is even, and therefore zero, since they intersect at most once. This is because $\tilde{\gamma}$ divides $\mathbb{H}^2$ into two separate connected components; and since $\tilde{C}$ and $\tilde{\gamma}$ have the same endpoints, if $\tilde{C}$ crosses $\tilde{\gamma}'$ it must cross it back again. Therefore, the intersection $\tilde{C} \cap \tilde{\gamma}$ is empty.

Since $C$ and $\gamma$ are homotopic, they are homologous, i.e. they cobound part of the surface; call it $N$. Moreover, the homotopy $H : S^1 \times [0, 1] \to F$ implies that $N$ is an annulus, assuming that $F$ is a homeomorphism on $N$. Since $H$ is continuous and $F$ is compact, it suffices to show that $H$ is injective; which follows from the fact that $C$ and $\gamma$ are homologous. Therefore, since $F$ is orientable, we get that $C$ and $\gamma$ are isotopic.

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8In Hebrew: לא משיקים
Bibliography


**Exercise**

Prove that theorem 2.2 (Any complete, connected, simply-connected hyperbolic surface is isometric to $\mathbb{H}^2$), implies that the universal cover of a compact hyperbolic surface is $\mathbb{H}^2$. 