Lemma 2.6: Suppose $C_1$ and $C_2$ are transverse essential $1$-submanifolds of a hyperbolic surface $F$ and no component of $C_1$ is isotopic to a component of $C_2$. Then $C_1$ and $C_2$ have minimal intersection if and only if there exists a homeomorphism $h: F \to F$, isotopic to the identity, such that $h(C_1)$ and $h(C_2)$ are both geodesic 1-submanifolds.

Remark: This fails for non-simple closed curves and for three or more 1-submanifolds, because of problems with triple points, see Figure 2.11.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure211.png}
\caption{Figure 2.11}
\end{figure}

Lemma 2.6 proof:

Let $C_1, C_2$ have all the characteristics in the beginning of the lemma.

\[ \Leftarrow \]

Let $h: F \to F$ be a homeomorphism isotopic to the $\text{Id}$ such that $h(C_1), h(C_2)$ are both geodesic 1-submanifolds. $h(C_1), h(C_2)$ have minimal intersection, indeed no component of the preimag of $h(C_1)$ in the universal cover $\mathbb{H}$ meets any component of the preimag of $h(C_2)$ more than once. Therefore $C_1, C_2$ have minimal intersection because
Isotopy doesn't change the number of intersections. (See exercise after the lemma).

First let's explain how we are going to prove it, before actually proving it.

Conversely, suppose $C_1$, $C_2$ have minimal intersection. By Lemma 2.4, we can assume that $C_2$ is geodesic. Again by lemma 2.4, $C_1$ is isotopic to a geodesic 1-submanifold $\gamma_1$. This isotopy can be achieved by a series of "pushes across disks" (Lemma 2.5) followed by a "push across an annulus". We shall choose this isotopy so that it leaves $C_2$ invariant.

We are going to assume $C_2$ is geodesic and move $C_1$ to $\gamma_1$ through isotopy that leaves $C_2$ invariant.

"pushes across disks"  "push across an annulus"

Figure 2.12
Now for the prove itself:
Suppose $A$ and $\alpha$ are arcs of $C_1, \gamma$, respectively $x$. $A \cup \alpha$ bounds an innermost disk $D$ (if they do not have such $D$ we can skip this part). So $C_2 \cap D$ is a family of arcs crossing $D$ from “top to bottom” (Figure 2.12) otherwise $C_2$ and $C_1$ (or $\delta$) will bound a disk contrary to them having minimal intersection. Now we can choose our “push across $D$” to leave $C_2$ invariant (although not pointwise fixed).

We will get that $A \cap \alpha = \emptyset$. By doing so again and again on finite many different disks we will get $C_1 \cap C_2 = \emptyset$.

Now we can move $C_1$ to $\delta$ via an isotopy which is the identity outside of an annulus $N$. As above, each component of $C_2 \cap N$ is an arc joining the two boundary components of $N$ and again this isotopy may be chosen to leave $C_2$ invariant.

Why is it called minimal intersection:

**Exercise:** Let $i(C_1,C_2) =$ the geometric intersection number of $C_1$ and $C_2$ be the minimum value of $|C_1 \cap C_2^\prime|$ where $C_1^\prime$ is homotopic to $C_1$. Prove that $C_1, C_2$ have minimal intersection if and only if $|C_1 \cap C_2| = i(C_1,C_2)$. 
Definition: Essential 1-submanifolds $C_1$, $C_2$ of $F$ fill $F$ if they have minimal intersection and every component of $F - (C_1 \cup C_2)$ is a disk.

Example:

Surface of genus 2. The surface is created by sticking 2 toruses each represented by a square. The sticking is made through a crack which is a parallel line in both squares.

Exercise: On any surface $F$, there exist simple closed curves $C_1$, $C_2$ filling $F$.

Theorem 2.7: Let $h: F \to F$ be an orientation preserving automorphism of a closed hyperbolic surface $F$. If for every essential simple closed curve $C$ in $F$ there is an integer $k > 0$ such that $h^k(C)$ is homotopic to $C$, then there exists an integer $n > 0$ such that $h^n$ is isotopic to the identity.

Proof: Choose geodesics $C_1$, $C_2$ filling $F$. There exist integers $k_1$, $k_2$ such that $h^{k_1}(C_1) \simeq C_1$ and $h^{k_2}(C_2) \simeq C_2$. Setting $k = k_1k_2$, $h^k(C_1) \simeq C_1$.

Note: We can choose such $C_1, C_2$ from lemma 2.6 since a filling remains a filling after isotopy.
for \( i = 1, 2 \). The curves \( h^k(c_1) \) and \( h^k(c_2) \) have minimal intersection, so by Lemma 2.6 there is a homeomorphism \( g \), isotopic to the identity, such that \( g \cdot h^k(c_1) \) is a closed geodesic for \( i = 1, 2 \). It follows that 
\[
g \cdot h^k(c_i) = c_i \text{ for } i = 1, 2, \text{ so } g \cdot h^k(c_1 \cap c_2) = c_1 \cap c_2.
\]

Considering \( c_1 \cup c_2 \) as a graph on \( F \), \( g \cdot h^k \) permutes the vertices of this graph, so there exists an \( m \geq 0 \) such that \( (g \cdot h^k)^m \) is isotopic to a homeomorphism \( f : F \to F \) which restricts to the identity on \( c_1 \cup c_2 \). As \( g \) is isotopic to the identity, it follows that \( h^{km} \) is isotopic to \( f \). As the complementary regions of \( c_1 \cup c_2 \) are disks, the Alexander trick shows that \( f \) is isotopic to the identity (see Figure 2.14).

\( \otimes \) This is the same proof as in the beginning of Lemma 2.6

\( \otimes \otimes \) This is from the uniqueness in Lemma 2.4.

**Alexander trick:**

Let \( h, g \) be a homeomorphisms of \( D^n \) which agree on the boundary surface \( S^{n-1} \) so \( h \) and \( g \) are isotopic.
The remainder of this chapter is devoted to the study of complete hyperbolic surfaces with finite area and geodesic boundary. One result is that there is a lower bound for the area of such surfaces. We begin by studying compact surfaces.

This topic is only for general knowledge and does not concern us in this seminar so proofs will not be shown.

Example: Choose a right rectangular geodesic hexagon in $\mathbb{H}^2$. Take a second copy and "glue" every other edge (see Figure 2.15). The resulting surface is the hyperbolic "pair of pants".

![Figure 2.15](image)

The area of this surface is $2(4\pi - 3\pi) = 2\pi$, $\chi_F = -1$.

Lemma 2.8: A compact hyperbolic surface with geodesic boundary has area $-2\pi \chi_F$.

Lemma proven in the proof of lemma 2.8:
Every closed hyperbolic surface is made by gluing the edges of a geodesic polygon in pairs.
For examples look at the notes of Eliav (in the first page).

Lemma 2.9: An unbounded complete hyperbolic surface with finite area is homeomorphic to a closed surface less a finite set and has area $-2\chi_F$.

Example: Consider an ideal triangle, $\text{Area} = \pi$. Double it to get a three punctured sphere, $\text{Area}=2\pi$.

![Figure 2.16](image)

Remark: If $F = \mathbb{H}^2/\Gamma$ is of finite area but non-compact, then $\Gamma$ has parabolic elements.

Theorem 2.10: A complete hyperbolic surface $F$ with finite area and geodesic boundary is homeomorphic to a compact surface less a finite set and has area $-2\pi\chi_F + \pi\chi_{\partial F}$.

Corollary 2.10.1: The area of any complete hyperbolic surface $F$ with totally geodesic boundary is $n\pi$, $n \geq 1$. In particular $\pi$ is a lower bound for such areas.
Exercise:
prove $|C_1 \cap C_2| = i(C_1, C_2) \Rightarrow C_1, C_2$ have minimal intersection.