Exercise Sheet, Diophantine Approximation, Fall 2021

Notations and assumptions. Vol (and sometimes Vol_n) denotes the Lebesgue measure on $\mathbb{R}^n$. Write $\mathbb{R}_+ = (0,\infty)$, and for $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, denote by φ-approx the set of $\theta \in \mathbb{R}$ for which there are infinitely many $p \in \mathbb{Z}$, $q \in \mathbb{N}$ such that $\left|\theta - \frac{p}{q}\right| < \varphi(q)$. The distance from $x \in \mathbb{R}$ to the nearest integer is denoted by $\langle x \rangle$ or by $\mathrm{dist}(x,\mathbb{Z})$. Write $x = \lfloor x \rfloor + \{x\}$ for the decomposition of $x \in \mathbb{R}$ to its fractional part and integer part, that is $\lfloor x \rfloor \in \mathbb{Z}$ and $\{x\} \in [0,1)$. The Hausdorff dimension of a set A is denoted by $\dim(A)$. The standard inner product of two vector $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is denoted by $\langle \mathbf{x}, \mathbf{y} \rangle$.

1. Let $\|\cdot\|$ and $\|\cdot\|'$ denote norms on $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Let

$$V = \mathrm{Vol}_m\left(\{x \in \mathbb{R}^m : \|x\| \le 1/2\}\right) \cdot \mathrm{Vol}_n\left(\{y \in \mathbb{R}^n : \|y\|' \le 1/2\}\right).$$

Show that there is $Q_0 > 0$ such that for all $X \in M_{m,n}(\mathbb{R})$ and any $Q > Q_0$, there are $\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}$, $\mathbf{p} \in \mathbb{Z}^m$ such that $\|\mathbf{q}\|' \le Q$ and $\|X\mathbf{q} - \mathbf{p}\| < V^{-1/m} Q^{-n/m}$. Give an explicit formula for Q_0, and prove your formula is optimal.

2. Let $\mathrm{GL}_2(\mathbb{Z})$ and $\mathrm{SL}_2(\mathbb{Z})$ denote respectively the group of integer 2×2 matrices with integer coefficients and determinant ± 1 (resp., determinant $+1$). Let

$$\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Prove that $\{\alpha^{\pm 1}, \beta^{\pm 1}\}$ generate $\mathrm{GL}_2(\mathbb{Z})$, and that $\{\alpha^{\pm 1}, \gamma^{\pm 1}\}$ generate $\mathrm{SL}_2(\mathbb{Z})$.

3. Let $d \in \mathbb{N}$, $d \ge 2$. Fix a norm $\|\cdot\|$ in $\mathbb{R}^d$. For $\theta \in \mathbb{R}^d$ we say that $(\mathbf{p}, q) \in \mathbb{Z}^d \times \mathbb{N}$ is a *best approximation* if for all $\mathbf{p}' \in \mathbb{Z}^d, q' \in \mathbb{N}$, with $q' \le q$ and $(\mathbf{p}', q') \ne (\mathbf{p}, q)$, we have $\|q\theta - \mathbf{p}\| < \|q'\theta - \mathbf{p}'\|$. Let $(\mathbf{p}_j, q_j)$ denote the sequence of best approximations for θ, ordered so that $q_1 \le q_2 \le \cdots$.

- Characterize the set of θ for which the sequence of best approximations is finite.
- Show that there is $C > 0$ (depending on d and the norm, but not on θ), such that for all j, $q_j^{1/d}\|q_j\theta - \mathbf{p}_j\| < C$.
- Show that for any j, any θ, and any norm, the vectors $\mathbf{u}_j = (\mathbf{p}_j, q_j), \mathbf{u}_{j+1} = (\mathbf{p}_{j+1}, q_{j+1})$ are a *primitive pair*; that is, they are linearly independent and any vector in $\mathbb{Z}^{d+1} \cap \mathrm{span}(\mathbf{u}_j, \mathbf{u}_{j+1})$ can be written as a linear combination of $\mathbf{u}_j, \mathbf{u}_{j+1}$ with integer coefficients.

- Give an example showing that three successive best approximations need not be linearly independent.

4. With the notation of Question 3, let $n = d + 1$, and let $\mathcal{L}_n = \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$ denote the space of n-dimensional lattices of covolume one. Let $a_t = \mathrm{diag}(e^t, \dots, e^t, e^{-dt})$ and let $\mathcal{S}$ denote the set of $\Lambda \in \mathcal{L}_n$ such that Λ contains a vector $\mathbf{x} = (x_1, \dots, x_d, 1)$ which is primitive, and the set

$$\{(y_1, \dots, y_n) : \|(x_1, \dots, x_d)\| \le \|(y_1, \dots, y_d)\|, |y_n| \le 1\}$$

does not intersect $\Lambda \smallsetminus \{0, \pm\mathbf{x}\}$. For $\theta = (\theta_1, \dots, \theta_d) \in \mathbb{R}^d$, let

$$\Lambda_\theta \stackrel{\mathrm{def}}{=} \{(p_1 - q\theta_1, \dots, p_d - q\theta_d, q) : p_1, \dots, p_d, q \in \mathbb{Z}\}.$$

Show that for each θ, the best approximations $(\mathbf{p}_j, q_j)$ and the numbers $t_j \stackrel{\mathrm{def}}{=} \frac{1}{d} \log q_j$ satisfy $\{t_j\} = \{t > 0 : a_t\Lambda_\theta \in \mathcal{S}\}$, and that $a_{t_j}\Lambda_\theta$ contains a primitive vector $\left(q_j^{1/d}(q_j\theta_1 - p_1), \dots, q_j^{1/d}(q_j\theta_d - p_d), 1\right)$, where $\mathbf{p}_j = (p_1, \dots, p_d)$.

5. Let

$$\mathcal{F}_n \stackrel{\mathrm{def}}{=} \left\{\frac{p}{q} : q \in \mathbb{N},\ p \in \{0, \dots, q\},\ \gcd(p, q) = 1,\ q \le n\right\}$$

denote the Farey set of level n.

- Prove that $\lim_{n\to\infty} \frac{\#\mathcal{F}_n}{n^2} = \frac{3}{\pi^2}$.
- Suppose that $\frac{p}{q} \in \mathcal{F}_n \smallsetminus \mathcal{F}_{n-1}$, and let $\frac{p'}{q'} < \frac{p}{q} < \frac{p''}{q''}$ be the neighbors of $\frac{p}{q}$ in $\mathcal{F}_n$, when the elements of $\mathcal{F}_n$ are given in increasing order. Suppose that $\frac{p}{q} = [a_1, \dots, a_N]$ (as a finite continued fraction). What are the continued fraction expansions of $\frac{p'}{q'}, \frac{p''}{q''}$?
- Suppose $\frac{p'}{q'} < \frac{p}{q}$ are two neighbors in $\mathcal{F}_n$. Prove that $\frac{p'+p}{q'+q} \in \mathcal{F}_{n_1}$ for some $n_1 > n$.

6. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that $x \mapsto x^2\varphi(x)$ is non-increasing, and $\lim_{x\to\infty} x^2\varphi(x) = 0$. For $c > 0$, denote $\varphi_c(x) = c\varphi(x)$. Show that there are uncountably many $\theta \in \mathbb{R}$ such that $\theta \in \varphi$-approx, but for any $c \in (0, 1)$, $\theta \notin \varphi_c$-approx.

7. Suppose $\psi : \mathbb{N} \to \mathbb{R}_+$ is a non-increasing function satisfying $\sum_{q=1}^{\infty} \psi(q) = \infty$. Show that there is non-increasing function $\bar{\psi} : \mathbb{N} \to \mathbb{R}_+$ such that $\sum_{q=1}^{\infty} \bar{\psi}(q) = \infty$ and for all $b \in \mathbb{N}$, $\lim_{q\to\infty} \frac{\bar{\psi}(q)}{\psi(bq)} = 0$.

8. Let

$$\beta_n \stackrel{\mathrm{def}}{=} \begin{cases} \frac{1}{n} & \text{if } n \text{ is prime} \\ 0 & \text{otherwise} \end{cases}$$

and let

$$\gamma_\ell \stackrel{\mathrm{def}}{=} \begin{cases} \frac{1}{k} & \text{if } \ell = 10^k, \quad \text{where } k \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Show that for a.e. $x \in \mathbb{R}$ (with respect to Lebesgue measure), there are infinitely many n for which $\langle nx \rangle < \beta_n$ and infinitely many ℓ for which $\langle \ell x \rangle < \gamma_\ell$.

9. Let m, n be positive integers with $n \geq 2$. Suppose $\varphi : \mathbb{N} \to [0, \infty)$ is non-increasing and $\sum_{r=1}^{\infty} r^{n-1} \varphi(r)^m = \infty$. Let $\widehat{\mathbb{Z}^n}$ denote the set of $\mathbf{q} = (q_1, \ldots, q_n) \in \mathbb{Z}^n$ which are primitive and satisfy $q_1 \geq 1$. Prove that $\sum_{\mathbf{q} \in \widehat{\mathbb{Z}^n}} \varphi(\|\mathbf{q}\|)^m = \infty$.

10. A number $x \in \mathbb{R}$ is called a *Liouville number* if for any $N > 0$ there are infinitely many distinct rationals $\frac{p}{q}$ such that $\left| x - \frac{p}{q} \right| < \frac{1}{q^N}$. A continuous bijection $h : \mathbb{R} \to \mathbb{R}$ with continuous inverse is called a *homeomorphism of* $\mathbb{R}$. Show that for any collection $h_1, h_2, \ldots$ of homeomorphisms of $\mathbb{R}$, there is $x \in \mathbb{R}$, such that the numbers $h_1(x), h_2(x), \ldots$ are all Liouville numbers. Show that there is $x \in \mathbb{R}$ such that x^k is a Lioville number for any $k \in \mathbb{N}$.

11. For $c > 0$, let

$$\mathrm{BA}_c \stackrel{\mathrm{def}}{=} \left\{ x \in \mathbb{R} : \forall (q, p) \in \mathbb{N} \times \mathbb{Z}, \; \left| x - \frac{p}{q} \right| \geq \frac{c}{q^2} \right\}.$$

Prove that for every $c > 0$, $\dim(\mathrm{BA}_c) < 1$.

12. Let $\mathrm{BA} = \bigcup_{c>0} \mathrm{BA}_c$. Say that a finite compactly supported measure μ on $\mathbb{R}$ is α-decaying if there is $C > 0$ such that for all $x \in \mathbb{R}$ and all $\varepsilon, r \in (0, 1)$,

$$\mu(B(x, \varepsilon r)) \leq C \varepsilon^\alpha \mu(B(x, r)).$$

Prove that there is an α-decaying probability measure μ on $\mathbb{R}$, such that $\mu(\mathrm{BA}) = 1$.

13. Recall that a measure μ on $\mathbb{R}^d$ is α-*absolutely decaying* if there are positive C, D such that for all $x \in \operatorname{supp} \mu$ and all $r \in (0, 1)$,

$$\mu(B(x, r)) \geq D \mu(B(x, 3r))$$

and for any $x \in \operatorname{supp} \mu$, any $0 < \varepsilon \leq r \leq 1$, and any affine hyperplane L,

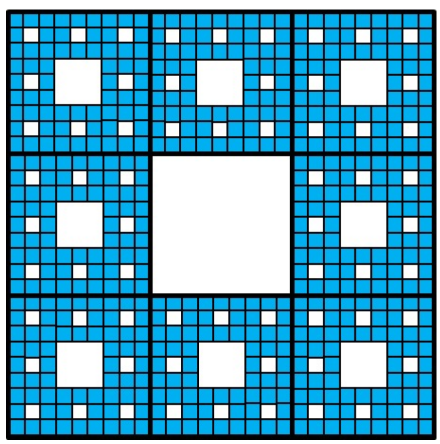
$$\mu(L^{(\varepsilon)} \cap B(x, r)) \leq C \left(\frac{\varepsilon}{r} \right)^\alpha \mu(B(x, r)), \quad \text{where } L^{(\varepsilon)} \stackrel{\mathrm{def}}{=} \bigcup_{x \in L} B(x, \varepsilon).$$

For $i = 1, 2$, let $d_i \in \mathbb{N}$ and let μ_i be a finite compactly supported measure on $\mathbb{R}^{d_i}$ which is α_i-absolutely decaying. Prove that $\mu_1 \times \mu_2$ is $\min(\alpha_1, \alpha_2)$-absolutely decaying on $\mathbb{R}^{d_1+d_2}$.

14. Define the *Sierpinski carpet* $\mathcal{S}$ as follows. Let $K_0 \stackrel{\text{def}}{=} [0,1]^2$ and for $(i,j) \in \mathcal{I} \stackrel{\text{def}}{=} \{0,1,2\}^2 \smallsetminus \{(1,1)\}$, let

$$T_{(i,j)} : K_0 \to K_0, \quad T_{(i,j)}(x) = \frac{x}{3} + \left(\frac{i}{3}, \frac{j}{3}\right).$$

Define K_n inductively as follows. Each K_n is a union of 8^n closed squares of sidelengths 3^{-n}. If $S_i^{(n)}$ is one of the squares comprising K_n, and $S_i^{(n)} = T(K_0)$ for an affine map $T : \mathbb{R}^2 \to \mathbb{R}^2$ of the form $T(x) = 3^{-n}(x) + y$, then the squares $T \circ T_{(i,j)}(K_0)$, where $(i,j) \in \mathcal{I}$, are 8 of the squares comprising K_{n+1}. Here is a picture of K_3.



Finally let $\mathcal{S} = \bigcap_{n=0}^{\infty} K_n$.

- Show that $\mathcal{S}$ is compact, satisfies
$$\mathcal{S} = \bigcup_{(i,j)\in\mathcal{I}} T_{(i,j)}(\mathcal{S}), \tag{0.1}$$
and is the unique non-empty compact subset of $\mathbb{R}^2$ satisfying (0.1).
- Show that there are uncountably many $x = (x_1, x_2) \in \mathcal{S}$ such that x_1, x_2 are both Liouville numbers.
- Show that $\dim(\mathcal{S}) = \frac{\log 8}{\log 3}$.
- Let $\mu_{\mathcal{S}}$ be the restriction to $\mathcal{S}$, of the $\dim(\mathcal{S})$-dimensional Hausdorff measure. Show that $\mu_{\mathcal{S}}$ is a finite and positive measure.
- Show that $\mu_{\mathcal{S}}$ satisfies $\mu_{\mathcal{S}} = \frac{1}{8}\sum_{(i,j)\in\mathcal{I}} (T_{(i,j)})_* \mu_{\mathcal{S}}$.
- Show that VWA has measure zero with respect to this measure.

15. Let $d \geq 2$, let $I = [0,1]$, and let Vol denote the restriction of Lebesgue measure on $\mathbb{R}$ to I. Let $T : I \to I$ be the map $T(x) = \{dx\}$ and let $J_n \subset I$, $n \in \mathbb{N}$ be a sequence of intervals such that $\Phi(N) \stackrel{\mathrm{def}}{=} \sum_{n=1}^{N} \mathrm{Vol}(J_n) \to_{N\to\infty} \infty$. Assume $\Phi(N) \to_{N\to\infty} \infty$. Show that for Vol-a.e. $x \in I$, and for any $\varepsilon > 0$,

$$\{1 \leq n \leq N : T^n(x) \in J_n\} = \Phi(N) + O\left(\Phi(N)^{1/2+\varepsilon}\right),$$

as $N \to \infty$.

16. Suppose M, ε are positive real numbers and $d \in \mathbb{N}$, with $M \geq \frac{2^d}{\varepsilon^d}$. Let $\mathbb{T}^d \stackrel{\mathrm{def}}{=} \mathbb{R}^d/\mathbb{Z}^d$, equipped with the metric

$$d(\pi(x), \pi(y)) \stackrel{\mathrm{def}}{=} \min_{p \in \mathbb{Z}^d} \|x - (y+p)\|_\infty,$$

where $x, y \in \mathbb{R}^d$ and $\pi : \mathbb{R}^d \to \mathbb{T}^d$ is the projection. Let

$$C_d(\varepsilon, M) \stackrel{\mathrm{def}}{=} \left\{\theta \in \mathbb{T}^d : (m\theta)_{0 \leq m \leq M} \text{ is not } \varepsilon\text{-dense in } \mathbb{T}^d\right\}.$$

Show that

$$C_d(\varepsilon, M) \subset \bigcup_{\mathbf{p}/q \in S} B\left(\frac{\mathbf{p}}{q}, \frac{1}{qM^{1/d}}\right),$$

where S is the set of all rational vectors $\mathbf{p}/q \in \mathbb{T}^d$ such that

$$1 \leq q \leq M \text{ and } \frac{\mathbf{p}}{q} \in C_d\left(\frac{\varepsilon}{2}, q\right).$$

17. Let $\eta_1, \eta_2, \ldots,$ be a sequence of real numbers. For any $\tau \geq 2$, and any $\eta \in \mathbb{R}$, let

$$A(\tau, \eta) = \left\{x \in \mathbb{R} : \text{ for infinitely many } (p,q) \in \mathbb{Z} \times \mathbb{N}, \left|x - \frac{p}{q} - \eta\right| < \frac{1}{q^\tau}\right\}.$$

Compute the Hausdorff dimension of the sets $\mathcal{A}(\tau, \eta_1)$, $A(\tau, \eta_1) \cap A(\tau, \eta_2)$, $\bigcap_i A(\tau, \eta_i)$.

18. For (p,q) a primitive vector in $\mathbb{Z}^2$, the *shortest solution of the gcd equation for* (p,q) is the vector (k, ℓ) of minimal Euclidean norm, satisfying $kp + \ell q = 1$, and the *signed length ratio* between two nonzero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ is

$$\mathrm{LR}(\mathbf{x}, \mathbf{y}) \stackrel{\mathrm{def}}{=} \pm \frac{\|\mathbf{y}\|}{\|\mathbf{x}\|},$$

where the sign is taken to be positive if and only if the angle between $\mathbf{x}$ and $\mathbf{y}$ is in $[0, \pi/2)$. Also let

$$\rho(\mathbf{x}, \mathbf{y}) \stackrel{\mathrm{def}}{=} \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\|^2}.$$

- Suppose $\mathbf{x} = (p, q) \in \mathbb{Z}^2$ is primitive and (k, ℓ) is the shortest solution of the gcd equation for (p, q). Write $\mathbf{y}_{\mathbf{x}} \stackrel{\mathrm{def}}{=} (-\ell, k)$. Show that the vectors $\mathbf{x}, \mathbf{y}_{\mathbf{x}}$ are a basis for the lattice $\mathbb{Z}^2$.
- Show that as $\|\mathbf{x}\| \to \infty$, for $\mathbf{x} \in \mathbb{Z}^2$ primitive,
$$|\mathrm{LR}(\mathbf{x}, \mathbf{y}_{\mathbf{x}}) - \rho(\mathbf{x}, \mathbf{y}_{\mathbf{x}})| \to 0.$$
- Show that for any $\mathbf{x} \in \mathbb{Z}^2$ primitive, $\rho(\mathbf{x}, \mathbf{y}_{\mathbf{x}}) \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.
- Prove that there is a Borel probability measure ν on $I \stackrel{\mathrm{def}}{=} \left[-\frac{1}{2}, \frac{1}{2}\right]$, which is absolutely continuous with respect to the restriction of Lebesgue measure to I, such that for any interval $J \subset I$, and almost every $x \in \mathbb{R}$, if $\mathbf{x}_k \stackrel{\mathrm{def}}{=} (p_k, q_k) \in \mathbb{Z}^2$ is such that $\gcd(p_k, q_k) = 1, q_k \geq 1$, and $\frac{p_k}{q_k}$ are the continued fraction convergents of x, and $\mathbf{y}_k \stackrel{\mathrm{def}}{=} \mathbf{y}_{\mathbf{x}_k}$ then
$$\frac{1}{N}\#\{1 \leq k \leq N : \rho(\mathbf{x}_k, \mathbf{y}_k) \in J\} \to_{N\to\infty} \nu(J).$$
- Write down a formula for the measure ν.