Corrections and additions to lecture of Dec. 14.

Prop $\mathcal{C} = \bigcap_{j=1}^{\infty} \bigcup_{I \in \mathcal{E}_j} I$, where each $\mathcal{E}_j$ is a collection of disjoint closed intervals, $D(I,J) \geq \varepsilon_j$ for $I, J \in \mathcal{E}_j$ distinct for a sequence $\varepsilon_j \searrow 0$, and each $I \in \mathcal{E}_{j-1}$ has at least m_j children in $\mathcal{E}_j$, where $m_j \geq 2$.

Then
$$\dim \mathcal{C} \geq \liminf_{j \to \infty} \frac{\log(m_1 \cdots m_{j-1})}{-\log(m_j \varepsilon_j)}$$

Pf A correct proof was given in the lecture, but the indexation in the statement (def. of m_j) was incorrect.

Lemma Let $\{\alpha_1, \alpha_2, \dots\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \dots\}$ (in the obvious order above). There is $c_1 > 0$ s.t. for any interval $I \subset [0,1]$, there is $k_0 = k_0(I)$ such that for any $k \geq k_0$ there is $t \in \mathbb{N}$ and indices $1 \leq i_1 < \dots < i_t$ s.t.: ⓐ $i_t \leq k$;

ⓑ $|\alpha_{i_r} - \alpha_{i_s}| \geq \frac{1}{k}$ for $r \neq s$; ⓒ $c_1 \operatorname{diam}(I) k \leq t$

ⓓ $t \leq \operatorname{diam}(I) k$ ⓔ $\alpha_j = \frac{p_j}{q_j}$ satisfies $q_j \leq \sqrt{3(j+1)} + 1$

ⓕ $\alpha_{i_j} \in I$.

$\psi_\tau = \frac{1}{q_\tau}$, ψ-approx $= \{x \in \mathbb{R} : \text{for inf. many } \frac{p}{q} \in \mathbb{Q},\ |x - \frac{p}{q}| < \psi(q)\}$

$= \limsup_{q \to \infty} B(\frac{p}{q}, \psi(q))$.

<u>Thm</u> (lower bound in $\frac{2}{\tau}$ theorem) For $\tau > 2$,

$\dim(\psi_\tau\text{-approx}) \geq \frac{2}{\tau}$.

<u>Pf</u>: Let $\mathcal{E}_0 = \{[0,1]\}$. We will inductively construct $\mathcal{E}_j$ as in the proposition, so that $\mathcal{C} = \bigcap_j \bigcup_{I \in \mathcal{E}_j} I$ satisfies $\mathcal{C} \subset \psi_\tau$-approx, and we have a lower bound for $\dim(\mathcal{C})$.

Let $\tau_0 > \frac{\tau}{2} > 1$, we will construct $\mathcal{C}$ so that $\dim(\mathcal{C}) \geq \frac{1}{\tau_0}$.

First choose $k_1, t_1, A_1 \subset \mathbb{N}$ s.t. the following hold: $\# A_1 = t_1$, $c_1 k_1 \leq t_1 \leq k_1$ for c_1 as in the Lemma, $A_1 \subset \{1, \ldots, k_1\}$, $\alpha_i \in [0,1]$ for $i \in A_1$, $|\alpha_i - \alpha_j| \geq \frac{1}{k_1}$ for $i \neq j$, $i, j \in A_1$. Such a choice is possible by the Lemma (with $I = [0,1]$).

Now suppose we have chosen $\mathcal{E}_i, A_i, k_i, t_i$ for $i = 1, \ldots, j$, where $\mathcal{E}_i = \{B(\alpha_i, \frac{1}{3 k_i^{\tau_0}}) : i \in A_i\}$, satisfying:

$\# A_i = t_i, \quad c_1 k_i \le t_i \le k_i, \; A_i \subset \{1, \dots, k_i\}$

for all i, and for all $j \in A_i$ there is $l \in A_{i-1}$

so that $B(\alpha_j, \frac{1}{3 k_i^{\tau_0}}) \subset B(\alpha_l, \frac{1}{3 k_{i-1}^{\tau_0}})$,

and $|\alpha_j - \alpha_l| \ge \frac{1}{k_i}$ for all $j \ne l$, $j, l \in A_i$.

The Lemma implies that we can make such choices. Namely, suppose $\mathcal{E}_j, A_j, k_j$ have been chosen. The Lemma shows that for each $i \in A_j$ setting $U = U_i = B(\alpha_i, \frac{1}{3 k_i^{\tau_0}})$, there is $k_0(U)$ s.t. for all $k \ge k_0$ we can choose $A^{(U)}$ s.t. the correct bounds hold for $\alpha \in A^{(U)}$, and

moreover $\cancel{B(\alpha, \frac{1}{3k^{\tau_0}}) \subset B(\alpha_l, \frac{1}{3 k_j^{\tau_0}})}$

$$B(\alpha, \frac{1}{3k^{\tau_0}}) \subset U.$$

We will choose $k_{j+1} \ge \max\{k_0(U_i) : i \in A_j\}$.

k_{j+1} will satisfy a further bound, to be explained below (see (*)). Set $A_{j+1} = \bigcup_{\substack{U = U_i \\ i \in A_j}} A^{(U)}$, $\mathcal{E}_{j+1} = \{B(\alpha, \frac{1}{3 k_{j+1}^{\tau_0}}) : \alpha \in A_{j+1}\}$

Define $m_{j+1} = 2c_1 \frac{k_{j+1}}{3 k_j^{\tau_0}}$. By (c) of Lemma, U_i contains at least m_{j+1} elements of $\mathcal{E}_{j+1}$.

Some remarks about these choices: $\frac{1}{k_i^{\tau_0}} < \frac{1}{k_i}$. 4

Since $|\alpha_r - \alpha_s| \geq \frac{1}{k_i}$ for $r \neq s$, $r, s \in A_i$,

the sets $U_r = B(\alpha_i, \frac{1}{3k_i^{\tau_0}})$ satisfy

$$D(U_r, U_s) \geq \varepsilon_i \stackrel{\text{def}}{=} \frac{1}{3k_i}.$$

The set $C = \bigcap_j \bigcup_{E \in \mathcal{E}_j} E$ is contained in ψ_τ-approx,

since $\tau_0 > \frac{\tau}{2}$, $\alpha_i = \frac{p_i}{q_i} \in A_j$ satisfies

$$q_i \underset{(e)}{\leq} \sqrt{3(i+1)} + 1 \leq 2\sqrt{i} \underset{(a)}{\leq} 2\sqrt{k_j} \quad \text{imply}$$

$$\frac{1}{3k_j^{\tau_0}} < \frac{1}{q_i^{\tau}}$$ for all large enough j.

The proposition implies $\dim(C) \geq \liminf_{j \to \infty} \frac{\sum_{\ell=1}^{j-1} \log(m_{\ell+1})}{-\log(m_{j+1}\varepsilon_{j+1})}$

$$\geq \frac{\sum_{\ell=1}^{j-1} \log\left(c_1 \frac{k_{\ell+1}}{3k_\ell^{\tau_0}}\right)}{-\log\left(\frac{2c_1 k_{j+1}}{3k_j^{\tau_0}} \cdot \frac{1}{3k_{j+1}}\right)} = \frac{\sum_{\ell=1}^{j-2} \text{const} + \log(\text{const} \cdot k_j)}{\log(\text{const}\ k_j^{\tau_0})}$$

where in the last formula, "const" means "independent of k_j". ~~By choosing $\delta \ldots$ and~~

choose $\delta > 0$. By choosing k_j large enough (depending on $k_1, \dots, k_{j-1}$) we can ensure:

$$(*) \quad \underbrace{\sum_{\ell=1}^{j-2} \text{const}_\ell} + \log(\text{const} \cdot k_j) \geq \frac{\log(k_j)}{1+\delta}$$

Since the numerator in the preceding inequality is $\text{const} + \tau_0 \log k_j$, we obtain that by choosing k_j to satisfy $(*)$ we get $\dim \mathcal{C} \geq \dfrac{1}{(1+\delta)\tau_0}$. Since δ was arbitrary we get $\dim(\psi_\tau\text{-approx}) \geq \frac{1}{\tau_0}$ and since τ_0 was arbitrary we get the result.