Chapter 1

APPROXIMATION OF INDEPENDENT QUANTITIES

§1. Khintchine's Theorem

In this chapter we will consider rational approximations of almost all real numbers (in the sense of Lebesgue measure). The following theorem, due to A. Khintchine [30, 34], has played a decisive role in the development of the whole area of metric number theory considered here.

Theorem 1. *Let $f(x)$ be a positive continuous function of a positive argument x, and suppose the function $xf(x)$ is nonincreasing. Then the inequality*

$$\left|\alpha - \frac{p}{q}\right| < \frac{f(q)}{q} \tag{1}$$

has an infinite set of solutions in integers p, $q > 0$ for almost all real numbers α, provided that the integral

$$\int_c^\infty f(x)\,dx \tag{2}$$

diverges for some $c > 0$. On the other hand, if the integral (2) converges, the inequality (1) has no more than a finite number of solutions in integers p, q for almost all α.

A. Khintchine deduced Theorem 1 from the theory of continued fractions. Let $[a_0; a_1, a_2, \ldots]$ be the continued fraction expansion of α, and let q_n be the denominator of the nth convergent of the fraction. Before Khintchine's work, it was known [8] that if $\psi(n)$ is an arbitrary positive function of a positive integer argument n, then the inequality $a_n = a_n(\alpha) \geqslant \psi(n)$ holds infinitely often for almost all α if the series $\sum 1/\psi(n)$ diverges, and holds only a finite number of times if the series converges. Khintchine supplemented this theorem with a metric estimate of the growth of $q_n = q_n(\alpha)$, namely, for almost all α

$$q_n(\alpha) < e^{Bn} \qquad (n > n_0(\alpha)),$$

where $B > 0$ is an absolute constant. It is then an easy matter to prove Theorem 1. A detailed proof of Theorem 1 can be found in Khintchine's book [34], and will not be reproduced here.

We now turn our attention to some special features of Khintchine's theorem. From this convergence of the integral (2) and the monotonicity of the function $xf(x)$ we can easily conclude that the inequality (1) has only a finite number of solutions in integers p, $q > 0$. Indeed, this fact is an immediate consequence of the following simple and general proposition, often called the Borel-Cantelli lemma:

Lemma 1. *Let A_q $(q = 1, 2, \ldots)$ be a sequence of measurable sets in $\mathbf{R}^n$ such that*

$$\sum_{q=1}^{\infty} |A_q| < \infty. \tag{3}$$

Then the measure of the set of real numbers falling in infinitely many A_q equals zero.

In fact, any number falling in an infinite sequence of sets A_q is contained in every set of the form

$$\bigcup_{q=p}^{\infty} A_q \qquad (p = 1, 2, \ldots),$$

whose measure, because of (3), can be made arbitrarily small for sufficiently large p.

If we now take A_q to be the set of all real numbers of the interval $[0, 1)$ for which (1) holds for a given $q > 0$, we easily find that $|A_q| = 2f(q)$, and since the convergence of the integral (2) and the monotonicity of $xf(x)$ implies the convergence of the series

$$\sum_{q=1}^{\infty} f(q), \tag{4}$$

we then obtain the part of Khintchine's theorem involving the convergence of the integral (2). Our assumption that the number α belongs to the interval $[0, 1)$ constitutes no restriction, since we can decompose the entire real line into a countable set of intervals of the form $[m, m+1)$, where m is an integer, and the character of the rational approximations to α and $\alpha + m$ is one and the same. We also see that convergence of the series (4) is enough to imply that the inequality (1) has only a finite number of solutions p, q for almost all α, without making any assumptions about the monotonicity of the function $xf(x)$.

On the other hand, the part of the theorem about the infinite number of solutions of (1) in the case where the integral (2) diverges is not so simple, and constitutes the basic content of Khintchine's theorem. In this regard, it is interesting to note that by Hurwitz's theorem [13, 35], for every irrational number α there exists an infinite number of solutions of equation (1) with $f(q) = 1/\sqrt{5}\,q$, where the constant $1/\sqrt{5}$ cannot be improved (for example, for $\alpha = \sqrt{5}$). Khintchine's theorem asserts that this approximation can be improved for almost all numbers by choosing, for example, $f(q) = 1/q \ln q$, but approximation with the function $f(q) = 1/q(\ln q)^{1+\delta}$, $\delta > 0$, is only possible on a set of measure zero.

The preceding remark on the finite number of solutions of (1) in the case where the series (4) converges allows us to pose the following question: If the series (4) diverges, then won't the inequality (1) have an infinite number of solutions for almost all α, or at least for a set of numbers α of positive measure?

We will see below (in §2, Theorem 6) that the answer to this question is in the negative, and hence that the assumption of the monotonicity of the function $xf(x)$ cannot be dropped. However, it can be replaced by the assumption that the sequence of numbers $q^{\gamma}f(q)$ is monotonic for some fixed γ (see §2, Theorem 3). The study of the inequality (1), without assuming that the sequence $f(q)$ or $q^{\gamma}f(q)$ is monotonic, changes the character of the problem in an essential way. In fact, without assuming that the function $xf(x)$ is monotonic, and relating the question of whether the number of solutions of the inequality (1) is finite or infinite to the question of whether the series (4) is convergent or divergent, we can consider the solutions of (1) in integers $q = q_k > 0$ $(k = 1, 2, \ldots)$ forming a given increasing sequence, as related to the question of whether the series

$$\sum_{k=1}^{\infty} f(q_k), \tag{5}$$

is convergent or divergent, since we can choose $f(q) = 0$ if $q \neq q_k$. For example, it follows from Lemma 1 that the inequality (1) has only a finite number of solutions in integers p, $q = q_k > 0$ for almost all α, if the series (5) is convergent. In particular, each of the inequalities

$$|\alpha q^2 - p| < \frac{1}{q(\ln q)^{1+\delta}}, \quad |\alpha 2^k - p| < \frac{1}{k(\ln k)^{1+\delta}} \quad (\delta > 0)$$

has only a finite number of solutions (in integers $p, q > 1, k > 1$) for almost all α.

Thus the attempt to free Theorem 1 from the condition that $xf(x)$ be monotonic compels us to consider more general "nonlinear" approximations:

$$|\alpha q_k - p| < f(q_k). \tag{6}$$

Here it is natural to assume that p and q_k in (6) are relatively prime numbers, $(p, q_k) = 1$, since otherwise the fraction p/q_k can be reduced to a new fraction and we cannot assert that the desired approximations are obtained with the given denominators q_k. In fact, we will study such approximations later in this chapter.

Finally, another peculiarity of Khintchine's theorem consists of the fact that its metric assertions have the character of assertions of the "almost all" or "almost none" type (the zero-one law). This phenomenon is a general feature of all the problems considered here.

We will obtain a proof of Theorem 1 as a consequence of Theorem 2 below, which deals with approximations of the form (6) in connection with the divergence of the series (5). Here we will make no use of the theory of continued fractions. This will allow us to use the same method to consider simultaneous approximations to a set of real numbers.

§2. The Duffin-Schaeffer Theorems

The following two theorems were proved by Duffin and Schaeffer [16]:

Theorem 2. *Let $f(q)$ $(q = 1, 2, \ldots)$ be an arbitrary sequence of nonnegative real numbers less than $1/2$ such that the series (4) diverges, and suppose there exists an infinite set of positive integers Q such that*

$$\sum_{q \leq Q} f(q) < c_1 \sum_{q \leq Q} f(q) \frac{\varphi(q)}{q}, \tag{7}$$

where $\varphi(q)$ is the Euler function and c_1 is a constant. Then for almost all α there exists an infinite number of solutions of the inequality (1) in integers $p, q > 0$ satisfying the condition $(p, q) = 1$.

The proof of this theorem is given in subsequent sections, and we now consider some consequences of the theorem. These consequences involve the replacement of the condition (7) by simpler assumptions about the behavior of the sequence $f(q)$, and can be deduced from

(7) by using easily obtained information on the distribution of the values of the Euler function $\varphi(q)$.

Lemma 2. *The asympotic formula*

$$\sum_{q\leqslant Q} \varphi(q) = c_2 Q^2 + O(Q \ln Q), \tag{8}$$

holds, where c_2 is an absolute constant.

PROOF. Denoting the Möbius function by $\mu(n)$, as usual, we have

$$\sum_{d \mid n} \mu(d) = \begin{cases} 1, & \text{if} \quad n = 1, \\ 0, & \text{if} \quad n \neq 1. \end{cases}$$

Therefore

$$\sum_{q\leqslant Q} \varphi(q) = \sum_{q\leqslant Q} \sum_{\substack{p\leqslant q \\ (p,q)=1}} 1 = \sum_{1\leqslant p\leqslant q\leqslant Q} \sum_{d \mid (p,q)} \mu(d) =$$
$$\sum_{d\leqslant Q} \mu(d) \sum_{p_1\leqslant q_1\leqslant Qd^{-1}} 1 = \sum_{d\leqslant Q} \mu(d) \frac{1}{2}\left(\left[\frac{Q}{d}\right]+1\right)\left[\frac{Q}{d}\right] =$$
$$\frac{1}{2} Q^2 \sum_{d\leqslant Q} \frac{\mu(d)}{d^2} + O\left(Q \sum_{d\leqslant Q} \frac{|\mu(d)|}{d}\right).$$

Since

$$\sum_{d\leqslant Q} \frac{\mu(d)}{d^2} = \frac{1}{\zeta(2)} + O\left(\sum_{d>Q} \frac{1}{d^2}\right) = \frac{1}{\zeta(2)} + O(Q^{-1}),$$
$$\sum_{d\leqslant Q} \frac{|\mu(d)|}{d} = O(\ln Q),$$

we obtain

$$\sum_{q\leqslant Q} \varphi(q) = \frac{1}{2\zeta(2)} Q^2 + O(Q \ln Q),$$

where $\zeta(2)$ is the value of the Riemann zeta function $\zeta(s)$ for $s = 2$, $\zeta(2) = \pi^2/6$. This proves formula (8).

Now let γ be an arbitrary number satisfying the condition $\gamma \leqslant 1$. Then, using Abel's transformation, from (8) we obtain

$$\sum_{q\leqslant Q} \frac{\varphi(q)}{q^{1+\gamma}} \gg \begin{cases} Q^{1-\gamma}, & \text{if} \quad \gamma < 1, \\ \ln Q, & \text{if} \quad \gamma = 1, \end{cases} \tag{9}$$

where the Vinogradov symbol $\gg$ allows for a quantity depending on γ. Similarly, if the sequence $q^\gamma f(q)$ is nonincreasing, then, writing the right-hand side of (7) in the form

$$\sum_{q\leqslant Q} q^\gamma f(q) \frac{\varphi(q)}{q^{1+\gamma}}$$

and making an Abel transformation, we deduce from (9) a relation of the form (7) and for all Q. This gives a stronger version of Theorem 1:

Theorem 3. *Let $f(q)$ $(q = 1, 2, \ldots)$ be an arbitrary sequence of nonnegative real numbers for which the series (4) diverges, and suppose there exists a $\gamma \leqslant 1$ such that $q^\gamma f(q)$ is nonincreasing. Then for almost all α there exists an infinite number of solutions of the inequality* (1) *in integers p, $q > 0$ satisfying the condition $(p, q) = 1$.*

As noted in the preceding section, by eliminating any assumptions about the monotonicity of $f(q)$ or $q^\gamma f(q)$, we can consider an inequality of the type (6) instead of the inequality (1) and use the divergence of the series (5) to prove that the number of solutions of (6) is infinite. In fact, if the sequence of increasing numbers q_k $(k = 1, 2, \ldots)$ is such that

$$\frac{\varphi(q_k)}{q_k} \geqslant \delta > 0 \qquad (k = 1, 2, \ldots) \tag{10}$$

for some fixed δ and the sequence $f(q)$ is such that $f(q) = 0$ for $q \neq q_k$, then the series (4) takes the form (5), and the condition (7) is replaced by the condition

$$\sum_{q_k \leqslant Q} f(q_k) < c_1 \sum_{q_k \leqslant Q} f(q_k) \frac{\varphi(q_k)}{q_k}, \tag{11}$$

which holds for all Q because of (10). Since

$$\frac{\varphi(q)}{q} = \prod_{p \mid q} \left(1 - \frac{1}{p}\right)$$

(where p runs over the prime divisors of q), the inequality (10) holds, for example, in the case where the q_k contain only a bounded number of prime divisors (the q_k are prime, say) or where all the prime divisors of q_k are bounded ($q_k = 2^k$). In these cases the divergence of the series (5) guarantees that (6) has an infinite number of solutions in integers p, q_k satisfying the condition $(p, q_k) = 1$.

If instead of (10) we can prove the weaker assertion

$$\sum_{k \leqslant N} \frac{\varphi(q_k)}{q_k} \geqslant \delta N \tag{12}$$

for some $\delta > 0$ and all N, then, assuming that $f(q_k)$ is a nonincreasing sequence, just as above we get (11) by making an Abel transformation in the right-hand side of (11). Formula (12) can be proved for many sequences q_k that do not satisfy (10). As an example, consider the case $q_k = P(k)$, where $P(x)$ is a polynomial with integer coefficients. In particular, if $P(x) = x^2$, then

$$\frac{\varphi(k^2)}{k^2} = \frac{\varphi(k)}{k} \qquad (k = 1, 2, \ldots),$$

so that (10) does not hold, but reasoning as in the proof of Lemma 2 we can obtain a formula stronger than (12).

Lemma 3. *Let $P(x)$ be a polynomial with integer coefficients whose leading coefficient is positive. Then*

$$\sum_{N_0 \leqslant k \leqslant N} \frac{\varphi(P(k))}{P(k)} = N \prod_{p \geqslant 2} \left(1 - \frac{r(p)}{p^2}\right) + O(N^{\varepsilon}), \tag{13}$$

where p runs over all prime numbers, $r(p)$ is the number of solutions of the congruence $P(k) \equiv 0 \pmod{p}$, $0 \leqslant k < p$, N_0 is the smallest integer satisfying the condition $P(k) > 0$ for $k \geqslant N_0$, and $\varepsilon > 0$ is arbitrary.

PROOF. We have

$$S_N = \sum_{N_0 \leqslant k \leqslant N} \frac{\varphi(P(k))}{P(k)} = \sum_{N_0 \leqslant k \leqslant N} \sum_{d \mid P(k)} \frac{\mu(d)}{d} = \sum_{d \leqslant P(N)} \frac{\mu(d)}{d} \sum_{\substack{k \leqslant N \\ P(k) \equiv 0 \pmod{d}}} 1 + O(1).$$

To calculate the sum

$$T_N(d) = \sum_{\substack{k \leqslant N \\ P(k) \equiv 0 \pmod{d}}} 1$$

for $d \leqslant N$, we divide the interval $[1, N]$ into intervals $[1, d]$, $[d+1, 2d]$, ... and possibly one interval of length less than d. If $r(d)$ is the number of solutions of the congruence $P(k) \equiv 0 \pmod{d}$, $0 \leqslant k < d$, then

$$T_N(d) = r(d)\left[\frac{N}{d}\right] + R_N(d),$$

for $d \leqslant N$, where $0 \leqslant R_N(d) \leqslant r(d)$. However if $d > N$, then $T_N(d) \leqslant r(d)$. Therefore

$$S_N = N \sum_{d \leqslant P(N)} \frac{\mu(d)\, r(d)}{d^2} + 2\theta \sum_{d \leqslant P(N)} \frac{|\mu(d)|}{d} r(d) + O(1),$$

where $0 \leqslant \theta \leqslant 1$. Since $\mu(d) = 0$ whenever d is divisible by the square of a prime number, we need only take account of square-free d. As is well known, the function $r(d)$ is multiplicative, so that $r(d) = r(p_1) \ldots r(p_s)$ if $d = p_1 \ldots p_s$, where $p_1, \ldots, p_s$ are distinct prime numbers. For prime numbers p that do not figure in the discriminant D_P of the polynomial $P(x)$ we have $r(p) \leqslant g$ if $D_P \neq 0$, where g is the degree of $P(x)$, and $r(p) \leqslant p$ in any case. Therefore, assuming first that $D_P \neq 0$, we find that

$$|\mu(d)|\, r(d) \leqslant |D_P|\, g^{\nu(d)}, \tag{14}$$

where $\nu(d)$ is the number of distinct prime divisors of d, and we get

$$\sum_{d \leqslant P(N)} \frac{|\mu(d)|}{d} r(d) \leqslant |D_P| \sum_{d \leqslant P(N)} \frac{g^{\nu(d)}}{d} = O(N^{\varepsilon}),$$

$$\sum_{d > P(N)} \frac{|\mu(d)|}{d^2} r(d) \leqslant |D_P| \sum_{d > P(N)} \frac{g^{\nu(d)}}{d^2} = O(P^{-1+\varepsilon}(N)),$$

if we take account of the fact that $g^{\nu(d)} = O(d^{\varepsilon})$. Therefore

$$S_N = N \sum_{d=1}^{\infty} \frac{\mu(d)\, r(d)}{d^2} + O(N^{\varepsilon}).$$

By the multiplicativity of $r(d)$ we obtain

$$\sum_{d=1}^{\infty} \frac{\mu(d)\, r(d)}{d^2} = \prod_p \left(1 - \frac{r(p)}{p^2}\right),$$

where the product, which extends over all prime numbers, is absolutely convergent and represents a quantity different from zero, since $r(p) \leqslant g$ for all sufficiently large p. In this way we get (13).

On the other hand, if $D_P = 0$, let $P = P_1^{\mu_1} \dots P_t^{\mu_t}$, where $P_1, \dots, P_t$ are distinct irreducible polynomials with integer coefficients. If $P(k)$ is divisible by a prime number p, then the product $P_0(k) = P_1(k) \dots P_t(k)$ is divisible by p, and conversely. Hence the number of those k for which $P(k) \equiv 0 \pmod p$, $0 \leqslant k < p$, coincides with the number of solutions of the congruence $P(k) \equiv 0 \pmod p$, $0 \leqslant k < p$. Since the discriminant of the polynomial $P_0(x)$ is different from zero, we again obtain an estimate of the form (14), in

which D_P is replaced by the discriminant of the polynomial $P_0(x)$.

Since (13) implies (12) for $q_k = P(k)$, we obtain

Theorem 4. *Let λ_k $(k = 1, 2, \dots)$ be an arbitrary nonincreasing sequence of nonnegative numbers, and let $P(x)$ be a polynomial with integer coefficients whose leading coefficient is positive. If the series*

$$\sum_{k=1}^{\infty} \lambda_k \tag{15}$$

diverges, then for almost all α the inequality

$$|\alpha P(k) - a| < \lambda_k$$

has an infinite number of solutions in integers $k > 0$ and a, subject to the condition $(P(k), a) = 1$.

In some cases (12) can be proved by using the fact that the relation (10) holds on a set of numbers k of positive asymptotic density. In this way, we can give another proof of (12) in the case $q_k = P(k)$ which is simpler than the proof based on the use of formula (13).

Lemma 4. *Let q_k $(k = 1, 2, \dots)$ be an arbitrary increasing sequence of positive integers such that*

$$\sum_{k \leqslant N} \sum_{p \mid q_k} \frac{1}{p} < c_3 N, \tag{16}$$

where p denotes the prime numbers and c_3 is a constant. Then for every δ in the interval

$$0 < \delta < e^{-c_3}\zeta^{-1}(2)$$

and for every integer $M > 0$, the number of those $k \leqslant M$ for which the inequality (10) holds is no less than βM, where

$$\beta = 1 - \frac{c_3}{\ln \frac{1}{\delta \zeta(2)}}. \tag{17}$$

PROOF. Let $\mathscr{N}\{k \leqslant M; \ldots\}$ denote the number of those $k \leqslant M$ satisfying the inequality appearing in the position of the dots. It follows from the condition (16) that

$$\mathscr{N}\left\{k \leqslant M;\ \sum_{p \mid q_k} \frac{1}{p} > \tau\right\} < \frac{c_3}{\tau} M$$

for every $\tau > 0$. Since

$$\ln \prod_{p \mid q_k}\left(1+\frac{1}{p}\right) < \sum_{p \mid q_k} \frac{1}{p},$$

we have

$$\mathscr{N}\left\{k \leqslant M;\ \prod_{p \mid q_k}\left(1+\frac{1}{p}\right) > T\right\} < \frac{c_3}{\ln T} M$$

for every $T > 1$. We now note that

$$\frac{q}{\varphi(q)} = \prod_{p \mid q}\left(1-\frac{1}{p}\right)^{-1} \leqslant \zeta(2) \prod_{p \mid q}\left(1+\frac{1}{p}\right),$$

since

$$\prod_{p \mid q}\left(1-\frac{1}{p}\right)^{-1} \prod_{p \mid q}\left(1+\frac{1}{p}\right)^{-1} = \prod_{p \mid q}\left(1-\frac{1}{p^2}\right)^{-1} \leqslant \prod_{p}\left(1-\frac{1}{p^2}\right)^{-1}.$$

But the last product is $\zeta(2)$. Therefore

$$\mathscr{N}\left\{k \leqslant M;\ \frac{q_k}{\varphi(q_k)} > T\zeta(2)\right\} < \frac{c_3}{\ln T} M,$$

or equivalently

$$\mathscr{N}\left\{k \leqslant M;\ \frac{\varphi(q_k)}{q_k} < \delta\right\} < \frac{c_3 M}{\ln \frac{1}{\delta\zeta(2)}},$$

where $\delta^{-1} = T\zeta(2)$. Hence

$$\mathscr{N}\left\{k \leqslant M;\ \frac{\varphi(q_k)}{q_k} \geqslant \delta\right\} \geqslant \beta M,$$

where β is given by formula (17), and this completes the proof of the lemma.

The condition (16) can be given the more transparent form

$$\sum_{p \leqslant q_N} \frac{1}{p} \mathscr{N}\{k \leqslant N;\ q_k \equiv 0 \pmod p\} < c_3 N.$$

For example, if

$$\mathscr{N}\{k \leqslant N;\ q_k \equiv 0 \pmod p\} \leqslant N\left(\frac{1}{\ln p} + \frac{1}{\ln\ln q_N}\right),$$

then the condition (16) holds, since

$$\sum_p \frac{1}{p \ln p} < \infty,$$

$$\sum_{p \leqslant q_N} \frac{1}{p} = \ln\ln q_N + O(1).$$

Summarizing what was said earlier, we obtain

Theorem 5. *If the sequence of positive integers q_k ($k = 1, 2, \ldots$) satisfies the condition (16), while the nonincreasing sequence of nonnegative numbers λ_k forms a divergent series (15), then the inequality*

$$|\alpha q_k - p| < \lambda_k$$

has an infinite number of solutions in integers p, q_k *satisfying the condition* $(p, q_k) = 1$.

The theorems of this section have all involved the case of an infinite number of solutions of inequalities of the form (1) or (6). As for the finiteness of the number of solutions, this is implied by the convergence of the series (4) or (5), as noted in the preceding section. However, since we have now been considering the solutions of (1) or (6) subject to the condition $(p, q) = 1$, the fact that the number of solutions is finite follows from the convergence of the series

$$\sum_{q=1}^{\infty} f(q) \frac{\varphi(q)}{q} \quad \text{or} \quad \sum_{k=1}^{\infty} f(q_k) \frac{\varphi(q_k)}{q_k}, \tag{18}$$

respectively, as is again obtained by using Lemma 1. Here it is possible for the series (4) or (5) to diverge, while the series (18) converges. In fact, let $q_k = p_1 p_2 \dots p_k$, where $p_1 = 2$, $p_2 = 3, \dots, p_k$ is the kth prime number in increasing order. Then

$$\frac{\varphi(q_k)}{q_k} = \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) < \frac{c_4}{\ln k}.$$

Choosing $f(q_k) = (k \ln k)^{-1}$, we get

$$\sum_{k \leqslant N} f(q_k) \sim \ln \ln N,$$

whereas

$$\sum_{2 \leqslant k \leqslant N} f(q_k) \frac{\varphi(q_k)}{q_k} < \sum_{2 \leqslant k \leqslant N} \frac{c_4}{k (\ln k)^2} < c_5.$$

Under the conditions of Khintchine's theorem, the convergence of the integral (2) and of the series (4) are equivalent, because of the monotonicity of $xf(x)$. Moreover, since the function $f(x)x^{-1}$ is nonincreasing, we can assume that $(p, q) = 1$ in the inequality (1), and if the number of solutions in integers p, q is infinite for some irrational number α, the same is true subject to the extra condition $(p, q) = 1$. Therefore, in the statement of Khintchine's theorem, we can add the condition $(p, q) = 1$. As already noted, under this condition the existence of an infinite number of solutions cannot be deduced from the divergence of the series (4) alone. We must therefore conclude that the assumption of monotonicity of $xf(x)$ in Khintchine's theorem is quite essential.

This phenomenon was observed by Duffin and Schaeffer [16], who also proved the following theorem, giving a negative answer to the question posed in the preceding section:

Theorem 6. *There exists a sequence of nonnegative numbers* $f(q)$ $(q = 1, 2, \dots)$ *for which the series* (4) *diverges, but such that the inequality* (1) *has an infinite number of solutions in integers* $p, q > 0$ *only for a set of numbers* α *of measure zero.*

We note that here it is not assumed that $(p, q) = 1$.

Since the infinite product $\prod_p \left(1 + \frac{1}{p}\right)$ taken over all the prime numbers diverges, we can find an infinite sequence of positive integers N_i $(i = 1, 2, \dots)$ such that $(N_i, N_j) = 1$ for $i \neq j$, N_i is divisible only by first powers of prime numbers, and

$$\prod_{p \mid N_i} \left(1 + \frac{1}{p}\right) > 2^{i+1} \qquad (i = 1, 2, \dots). \tag{19}$$

Let the numbers $f(q)$ be defined by the formula

$$f(q) = \begin{cases} 2^{-i-1} \frac{q}{N_i}, & \text{if } \quad q \mid N_i,\ q > 1, \\ 0, & \text{if } \quad q \nmid N_i \ \text{ or } \ q = 1. \end{cases}$$

and let A_q denote the set of points of the interval (0, 1) formed by the intervals

$$(0,\ f(q)/q), \quad (1 - f(q)/q,\ 1), \quad (p/q - f(q)/q,\ \ p/q + f(q)/q) \quad (p = 1, \dots, q-1).$$

Then $|A_q| = 2f(q) = 2^{-i}q/N_i$, if q is a divisor of N_i, $q > 1$. Considering the union of all the sets A_q for all divisors $q > 1$ of a fixed N_i, we see that this union coincides with A_{N_i},

$$\Big|\bigcup_{q \mid N_i} A_q\Big| = |A_{N_i}| = 2^{-i} \qquad (i = 1, 2, \ldots). \tag{20}$$

If now A is the set of all α in the interval (0, 1) for which there exists an infinite number of pairs q, $q > 0$ satisfying (1), then

$$A \subseteq \bigcup_{i=k}^{\infty} \bigcup_{q \mid N_i} A_q$$

for every k, and by (20),

$$|A| \leqslant \sum_{i=k}^{\infty} \Big|\bigcup_{q \mid N_i} A_q\Big| = 2^{-k+1},$$

so that $|A| = 0$. At the same time, by the definition of $f(q)$ we have

$$\sum_{q=1}^{\infty} f(q) = \sum_{i=1}^{\infty} 2^{-i-1} \frac{1}{N_i} \sum_{\substack{q \mid N_i \\ q > 1}} q = \infty,$$

since according to the inequality (19),

$$\frac{1}{N_i} \sum_{\substack{q \mid N_i \\ q > 1}} q = \frac{1}{N_i} \prod_{p \mid N_i} (1+p) - \frac{1}{N_i} > \prod_{p \mid N_i} \left(1 + \frac{1}{p}\right) - 1 \geqslant 2^i.$$

As for the behavior of the series (18), we find that

$$\sum_{q=1}^{\infty} f(q) \frac{\varphi(q)}{q} = \sum_{i=1}^{\infty} 2^{-i-1} \frac{1}{N_i} \sum_{\substack{q \mid N_i \\ q > 1}} \varphi(q) =$$

$$\sum_{i=1}^{\infty} 2^{-i-1} \frac{N_i - 1}{N_i} < 1,$$

since $\sum_{q \mid N} \varphi(q) = N$ by Euler's formula. This again shows that the behavior of the series (18) is more important than that of (4) or (5), as far as the solvability of (1) by infinitely many pairs p, q is concerned.

Duffin and Schaeffer [16] enunciated the hypothesis that the inequality (1) has an infinite number of solutions in integers p, $q > 0$ for almost all α if and only if the series

$$\sum_{q=1}^{\infty} f(q) \frac{\varphi(q)}{q}$$

diverges. This hypothesis has not yet been proved and probably involves delicate questions of the distribution of rational numbers on the real line. Recently Erdös [19] has made some progress with this problem (see §8).

§3. Proof of the Basic Theorem

To prove Theorem 2, as well as other similar assertions, we must establish that the measure of the set A of points falling in an infinite number of the measurable sets A_q ($q = 1, 2, \ldots$) is positive. We can then use arguments from ergodic theory to prove that the set A contains almost all numbers.

The following basic lemma [72], involving sequences A_q of measurable sets of an abstract space Ω with a finite measure μ, defined on the σ-algebra **A** of measurable subsets of Ω, is often useful as the first step. Our applications of this lemma will all pertain to the case where Ω is contained in the unit n-dimensional cube E^n equipped with Lebesgue measure.

Lemma 5. *Let* $(\Omega, \mathbf{A}, \mu)$ *be a measure space, and let* $A_q \in \mathbf{A}$ *be* a sequence of sets such that

$$\sum_{q=1}^{\infty} \mu A_q = \infty. \tag{21}$$

Then the set A of points falling in infinitely many sets A_q is of measure

$$\mu(A) \geqslant \overline{\lim_{Q \to \infty}} \frac{\left(\sum_{q=1}^{Q} \mu(A_q)\right)^2}{\sum_{p,\, q=1}^{Q} \mu(A_q \cap A_p)}. \tag{22}$$

PROOF. Given a pair of integers m, n $(1 \leqslant m \leqslant n)$, we set

$$A_m^n = \bigcup_{m \leqslant q \leqslant n} A_q, \quad A^m = \bigcup_{q \geqslant m} A_q.$$

Then $A_m^n \subseteq A^m$,

$$\mu(A) = \lim_{m \to \infty} \mu A^m \geqslant \lim_{m \to \infty} \left(\lim_{n \to \infty} \mu A_m^n\right). \tag{23}$$

We now estimate the measure of the sets A_m^n from below. Let $\omega \in \Omega$, and let $N_m^n(\omega)$ be the number of those A_q, $m \leqslant q \leqslant n$, which contain ω. The function $N_m^n(\omega)$ is obviously μ-measurable, since it can be represented in the form

$$N_m^n(\omega) = \sum_{m \leqslant q \leqslant n} \chi_q(\omega), \tag{24}$$

where $\chi_q(\omega)$ is the characteristic function of the set A_q. By the Cauchy-Schwarz inequality,

$$\left(\int_{\Omega} N_m^n(\omega)\, d\mu\right)^2 = \left(\int_{A_m^n} N_m^n(\omega)\, d\mu\right)^2 \leqslant$$

$$\int_{A_m^n} d\mu \int_{A_m^n} (N_m^n(\omega))^2\, d\mu = \mu A_m^n \int_{\Omega} (N_m^n(\omega))^2\, d\mu.$$

Therefore

$$\mu A_m^n \geqslant \frac{\left(\int_{\Omega} N_m^n(\omega)\, d\mu\right)^2}{\int_{\Omega} (N_m^n(\omega))^2\, d\mu}. \tag{25}$$

Using (24), we get

$$\int_{\Omega} N_m^n(\omega)\, d\mu = \sum_{m \leqslant q \leqslant n} \int_{\Omega} \chi_q(\omega)\, d\mu = \sum_{m \leqslant q \leqslant n} \mu A_q,$$

$$\int_{\Omega} (N_m^n(\omega))^2\, d\mu = \sum_{m \leqslant p,\, q \leqslant n} \int_{\Omega} \chi_p(\omega)\chi_q(\omega)\, d\mu = \sum_{m \leqslant p,\, q \leqslant n} \mu(A_q \cap A_p),$$

so that (25) takes the form

$$\mu A_m^n \geqslant \frac{\left(\sum_{m \leqslant q \leqslant n} \mu A_q\right)^2}{\sum_{m \leqslant p,\, q \leqslant n} \mu(A_p \cap A_q)}. \tag{26}$$

Next we note that

$$\sum_{m \leqslant q \leqslant n} \mu A_q = \sum_{1 \leqslant q \leqslant n} \mu A_q + O(1)$$

for fixed m, because of (21). Moreover

$$\sum_{m \leqslant p,\, q \leqslant n} \mu(A_p \cap A_q) =$$

$$\sum_{1 \leqslant p,\, q \leqslant n} \mu(A_p \cap A_q) + O\Big(\sum_{1 \leqslant q \leqslant n} \mu A_q\Big).$$

Since $A_m^n \subseteq \Omega$ and $\mu\Omega < \infty$, it follows from (26) that for fixed m

$$\sum_{m \leqslant p,\, q \leqslant n} \mu(A_p \cap A_q) \geqslant \frac{1}{\mu\Omega}\Big(\sum_{m \leqslant q \leqslant n} \mu A_q\Big)^2 \sim \frac{1}{\mu\Omega}\Big(\sum_{1 \leqslant q \leqslant n} \mu A_q\Big)^2$$

as $n \to \infty$. Therefore

$$\sum_{m \leqslant q \leqslant n} \mu A_q \sim \sum_{1 \leqslant q \leqslant n} \mu A_q,$$
$$\sum_{m \leqslant p,\, q \leqslant n} \mu(A_p \cap A_q) \sim \sum_{1 \leqslant p,\, q \leqslant n} \mu(A_p \cap A_q) \qquad (n \to \infty).$$

We now obtain (22) from (23) and (26).

Instead of (22), it is sometimes convenient to use an inequality of the same form, but one in which the summation of μA_q and of $\mu(A_q \cap A_p)$ is carried out for q between the limits $m \leqslant q \leqslant Q$ and p, q between the limits $m \leqslant p, q \leqslant Q$ for some fixed m. It is clear from the foregoing that we have such an inequality as well.

Direct application of Lemma 5 does not always allow us to prove that A contains almost all points of Ω, and then additional arguments are needed to show that this is so. In our specific case the problem is solved by the following theorem due to Gallagher [20]:

Theorem 7. *Let $f(q)$ $(q = 1, 2, \ldots)$ be an arbitrary sequence of nonnegative real numbers. Then the measure of the set A of numbers in the interval* $[0, 1)$ *for which the inequality* (1) *has infinitely many solutions in integers p, $q > 0$ satisfying the condition $(p, q) = 1$ is equal to either* 0 *or* 1.

We will first prove this theorem, and we will then use Lemma 5 to deduce from the conditions of Theorem 2 that $|A| > 0$. This will give the result of Theorem 2 for almost all numbers. Although

Theorem 2 can be proved by a direct and more elementary method (see [16]), we prefer to exhibit this more complicated method for the reason that it is based on ideas that are useful for the solution of other problems.

We preface the proof of Theorem 7 with two simple lemmas.

Lemma 6. *Let I_k $(k = 1, 2, \ldots)$ be a sequence of intervals, $|I_k| \to 0$ $(k \to \infty)$, and let A_k $(k = 1, 2, \ldots)$ be a sequence of measurable sets for which $A_k \subseteq I_k$ and*

$$|A_k| \geqslant \delta |I_k| \qquad (k = 1, 2, \ldots) \tag{27}$$

for some $\delta > 0$. Then the measure of the set of points falling in infinitely many I_k coincides with the measure of the set of points falling in infinitely many A_k.

PROOF. Let

$$J = \bigcap_{l=1}^{\infty} \bigcup_{k=l}^{\infty} I_k, \qquad B_l = \bigcup_{k=l}^{\infty} A_k \qquad D_k = J \setminus B_k.$$

It is enough to prove that every D_k is of measure zero. Suppose this is not so. Then by Lebesgue's theorem, D_k contains a point x_0 of metric density. Since $x_0 \in J$ for infinitely many k, we have $x_0 \in I_k$, and for such k

$$|D_k \cap I_k| \sim |I_k| \qquad (k \to \infty) \tag{28}$$

since $|I_k| \to 0$ $(k \to \infty)$. On the other hand, the sets D_k and A_k do not intersect if $k \geqslant l$, and hence $D_k \cap I_k$ and A_k are nonintersecting subsets of the interval I_k. Therefore

$$|I_k| \geqslant |A_k| + |D_k \cap I_k| \geqslant \delta |I_k| + |D_k \cap I_k|,$$
$$|D_k \cap I_k| \leqslant (1 - \delta)|I_k| \qquad (k \geqslant l),$$

which contradicts (28).

Lemma 7. *For every pair of integers $q \geqslant 2$ and s the transformation T of the interval* $[0, 1)$ *into itself given by the rule*

$$x \to qx + \frac{s}{q} \pmod 1$$

is ergodic, i.e., every invariant set is of measure 0 *or* 1.

PROOF. Let A be an invariant subset. Then A goes into itself, and $x \to q^n x + \frac{s}{q} \pmod 1$ under the nth iteration of the transformation T. Let $\chi(x)$ be the characteristic function of $A \pmod 1$. Then $\chi(x) \leqslant \chi\left(q^n x + \frac{s}{q}\right)$.

Suppose that $|A| > 0$. Then A contains a point x_0 of metric density. We choose an interval I_n of length q^{-n} centered at the point x_0. Then

$$|A \cap I_n| = \int_{I_n} \chi(x)\,dx \leqslant$$

$$\int_{I_n} \chi\left(q^n x + \frac{s}{q}\right) dx = \frac{1}{q^n} \int_0^1 \chi(x)\,dx = |I_n|\,|A|.$$

Since x_0 is a point of metric density of A, the left-hand side of the last formula is asymptotically equivalent to $|I_n|$ as $n \to \infty$. Therefore $|A| = 1$.

A more intuitive proof of Lemma 7 can be obtained by another argument. Almost all points of A are normal numbers in the sense of Borel, and if $|A| > 0$, there exists a point x_0 of metric density in A which is a normal number. Since all the numbers $q^n x_0 + \frac{s}{q} \pmod 1$ $(n = 1, 2, \ldots)$ belong to A along with x_0 and are points of metric density of A, while the sequence $q^n x_0 + \frac{s}{q} \pmod 1$ is uniformly distributed on the interval $[0, 1)$, it is obvious that $|A| = 1$.

We now turn to the proof of Theorem 7.

First of all, we note that we can exclude values of $f(q)$ that are too large. More exactly, it can be assumed that

$$f(q) < q^{\varepsilon_0} \qquad (q = 1, 2, \ldots),$$

where $0 < \varepsilon_0 < 1$. In fact, if there exists an infinite sequence q_k such that $f(q_k) \geqslant q_k^{\varepsilon_0}$, then for any α, we can find an integer p_k, corresponding to the given q_k, which satisfies the condition

$$\left|\alpha - \frac{p_k}{q_k}\right| < \frac{f(q_k)}{q_k}, \qquad (p_k, q_k) = 1.$$

This follows from the fact that for sufficiently large q there is an integer relatively prime to q in every interval of length q^{ε_0}. To see this, we introduce the function

$$\Phi_q(x) = \sum_{\substack{p < x \\ (p,\, q) = 1}} 1.$$

Reasoning as in the proof of Lemma 2, we get

$$\Phi_q(x) = \sum_{d \mid q} \mu(d) \sum_{p \leqslant x/d} 1 =$$

$$x \sum_{d \mid q} \frac{\mu(d)}{d} + O\left(2^{\nu(q)}\right) = x \frac{\varphi(q)}{q} + O(q^{\varepsilon}),$$

where $\varepsilon > 0$ is arbitrary. Since $\varphi(q) > q^{1-\varepsilon}$ for sufficiently large q, the number of integers which are relatively prime to q and lie in an interval of length q^{ε_0} equals

$$q^{\varepsilon_0} \frac{\varphi(q)}{q} + O(q^{\varepsilon}) > \frac{1}{2} q^{\varepsilon_0 - \varepsilon} > 1,$$

which implies the remark made above.

We now take the prime number p and an integer $n \geqslant 1$, and consider approximations to α of the form

$$\left|\alpha - \frac{a}{q}\right| < \frac{f(q)\, p^{n-1}}{q}, \qquad (a, q) = 1. \tag{29}$$

Let $A(p^n)$ and $B(p^n)$ be the sets of those α which satisfy an infinite number of inequalities (29) subject to the conditions $p \nmid q$ and $p \parallel q$ respectively. The sets $A(p^n)$ and $B(p^n)$ do not decrease as n increases, and $A(p)$ and $B(p)$ are contained in A. Applying Lemma 6, we find that

$$|A(p)| = |A(p^n)| \qquad (n = 1, 2, \ldots).$$

Hence the union $A^*(p)$ of all the $A(p^n)$ is of measure $|A(p)|$. If α satisfies (29) and $p \nmid q$, then, multiplying (29) by p, we see that the transformation $x \to px$ (mod 1) carries $A(p^n)$ into $A(p^{n+1})$, and therefore carries $A^*(p)$ into itself. But then, by Lemma 7, the measure of $A^*(p)$ is equal to 0 or 1, and hence $|A(p)|$ equals 0 or 1. The same argument shows that $|B(p)|$ equals 0 or 1, but here we consider the transformation $x \to px + \frac{1}{p}$ (mod 1): If α satisfies (29) and $p \parallel q$, then

$$\left| p\alpha + \frac{1}{p} - \frac{pa + \frac{q}{p}}{q} \right| < \frac{f(q)\, p^n}{q}, \qquad \left(pa + \frac{q}{p},\ a\right) = 1.$$

Thus, if at least one of the sets $A(p)$ and $B(p)$ has positive measure for some prime p, then $|A| = 1$ and the proof of Theorem 7 is complete.

Therefore we assume that $|A(p)| = |B(p)| = 0$ for all p. Now let $C(p)$ be the set of those α for which there exist infinitely many solutions of the inequality

$$\left| \alpha - \frac{a}{q} \right| < \frac{f(q)}{q}, \qquad (a, q) = 1, \tag{30}$$

satisfying the condition $p^2 \mid q$. It is obvious that

$$A = A(p) \cup B(p) \cup C(p), \qquad |A| = |C(p)|$$

for all prime numbers p. If α satisfies (30) with $p^2 \mid q$, then

$$\left| \alpha \pm \frac{1}{p} - \frac{a \pm \frac{q}{p}}{q} \right| < \frac{f(q)}{q}, \qquad \left(a \pm \frac{q}{p},\ q\right) = 1.$$

Hence $C(p)$ contains along with α all numbers of the form $\alpha + \frac{s}{p}$ (mod 1), where s is an arbitrary integer. Then for every interval $I(p)$ of length $1/p$ we have

$$|C(p) \cap I(p)| = |C(p)|\,|I(p)|,$$

and since A differs from $C(p)$ on a set of measure zero,

$$|A \cap I(p)| = |A|\,|I(p)|$$

for all prime numbers p. If we now assume that $|A| > 0$, then, taking a point of metric density in A and a sequence of intervals $I(p)$ of length $1/p$ centered at this point, we find that

$$|A \cap I(p)| \sim |I(p)| \qquad (p \to \infty).$$

Therefore $|A| = 1$, and the proof of Theorem 7 is complete.

We can now easily obtain the proof of Theorem 2.

Consider the set A_q of those α in the interval $[0, 1)$ which satisfy (30) for a given q. Then A_q is the union of $\varphi(q)$ open non-intersecting intervals, each of length $2f(q)/q$, and

$$|A_q| = 2f(q)\,\varphi(q)/q. \tag{31}$$

We now estimate the measure of the intersection of the sets A_{q_1} and A_q $(q_1 < q)$. Obviously

$$|A_{q_1} \cap A_q| \leqslant 2 \min\left(\frac{f(q_1)}{q_1}, \frac{f(q)}{q}\right) N(q_1, q), \tag{32}$$

where $N(q_1, q)$ is the number of pairs of integers a, a_1 for which the conditions

$$\left|\frac{a}{q}-\frac{a_1}{q_1}\right|<\frac{f(q)}{q}+\frac{f(q_1)}{q_1},$$
$$(a,\ q)=(a_1,\ q_1)=1,\quad 0<a<q,\quad 0<a_1<q_1, \tag{33}$$

hold for given q, q_1. If the equality

$$aq_1-a_1q=t \tag{34}$$

holds for some t, then $d=(q_1,\ q)$ divides t, and then, setting $q_1=dq_1'$, $q=dq'$, $t=dt'$, we have $aq_1'-a_1q'=t'$, $(q_1',\ q')=1$. If a', a_1' also satisfy (34), then the last fact implies

$$a=a'+kq',\quad a_1=a_1'+kq_1',\quad k \text{ an integer}. \tag{35}$$

Since we are interested in numbers a, a' lying in the interval $(0,\ q)$, the conditions

$$|a-a'|=|k|q'<q,\quad |k|<d,$$

must hold. Hence the number of possible values of a satisfying (34) for a given t is no greater than $2d-1$, and, because of (35), the same quantity estimates the number of admissible pairs a, a'.

Finally, since (33) implies

$$0\neq|t|<q_1f(q)+qf(q_1)$$

and we must only take values of t divisible by d, we find that

$$N(q_1,\ q)\leqslant 2\left[\frac{q_1f(q)+qf(q_1)}{d}\right](2d-1)<$$
$$4(q_1f(q)+qf(q_1)).$$

Then the inequality (32) gives

$$|A_{q_1}\cap A_q|\leqslant 16\,f(q_1)\,f(q). \tag{36}$$

Thus from (31) and the last inequality we get

$$\sum_{q\leqslant Q}|A_{q_1}\cap A_q|\leqslant 32\sum_{q_1<q\leqslant Q}f(q_1)f(q)+$$
$$2\sum_{q\leqslant Q}f(q)<17\Big(\sum_{q\leqslant Q}f(q)\Big)^2$$

for all sufficiently large Q, since the sum $\sum_{q\leqslant Q}f(q)$ is infinitely small compared to its square, because of the divergence of the series (4). The condition (7) and equation (31) now give

$$\sum_{q_1,\ q\leqslant Q}|A_{q_1}\cap A_q|<5c_1^2\Big(\sum_{q\leqslant Q}|A_q|\Big)^2$$

for infinitely many Q. But then $|A|>(5c_1^2)^{-1}$, by Lemma 5, and the proof is completed by applying Theorem 7.

We note that the condition $f(q)<1/2$ can be dropped in the statement of Theorem 2. We recommend that the reader prove this by himself as an exercise.

By estimating the measure of the intersection $A_{q_1}\cap A_q$ more carefully, we can obtain other theorems of the same type as Theorem 2. For example, it is not hard to show that

$$N(q_1,\ q)\leqslant 2\sum_{d\mid q,\ d'\mid q_1}\mu(d)\,\mu(d')\left[\frac{T}{dd'd_1}\right]d_1, \tag{37}$$

where $d_1=\left(\frac{q}{d},\ \frac{q_1}{d_1}\right)$, $T=q_1f(q)+qf(q_1)$. Therefore, making (37) substantially cruder, we have

$$N(q_1,\ q)\leqslant 2T\frac{\varphi(q_1)}{q_1}\frac{\varphi(q)}{q}+O(2^{\nu(q_1)+\nu(q)}(q_1,\ q)),$$

which gives

$$|A_{q_1}\cap A_q|\leqslant 4f(q_1)\,f(q)\frac{\varphi(q_1)}{q_1}\frac{\varphi(q)}{q}+$$
$$O\left(\min\left(\frac{f(q_1)}{q_1},\ \frac{f(q)}{q}\right)2^{\nu(q_1)+\nu(q)}(q_1,\ q)\right)$$

because of (32). This shows that the condition (7) in Theorem 2 can be replaced by the condition

$$\sum_{q\leqslant Q} f(q)\, 4^{\nu(q)}\tau(q) \ln q \ll c_6\left(\sum_{q\leqslant Q} f(q)\frac{\varphi(q)}{q}\right)^2.$$

Since

$$4^{\nu(q)}\tau(q)\ln q \ll q^{\varepsilon}, \qquad \frac{\varphi(q)}{q} \gg q^{-\varepsilon},$$

for arbitrary $\varepsilon > 0$, we see that the assertion of Theorem 2 is true if there are infinitely many integers Q such that

$$\sum_{q\leqslant Q} f(q) \geqslant Q^{\delta},$$

for some $\delta > 0$. Therefore sufficiently "rapid" divergence of the series (4) guarantees the existence of infinitely many solutions of the inequality (1) in relatively prime integers p, q for almost all α.

§4. Simultaneous Approximations

In reference [32] A. Khintchine gave a generalization of his theorem to the case of simultaneous approximations.

Theorem 8. *Let $n \geqslant 1$ be an integer, and let $f(x)$ be a positive continuous function of a positive argument x such that $xf^n(x)$ converges monotonically to zero as $x \to \infty$. Then the system of inequalities*

$$\max(|\alpha_1 q - p_1|, \ldots, |\alpha_n q - p_n|) < f(q) \tag{38}$$

has an infinite set of solutions in integers $q > 0$ for almost all n-tuples of real numbers $\alpha_1, \ldots, \alpha_n$, provided that the integral

$$\int_c^{\infty} f^n(x)\,dx \tag{39}$$

diverges for some $c > 0$. On the other hand, if the integral (39) *converges, the inequality* (39) *has no more than a finite number of solutions in integers $q > 0$, $p_1, \ldots, p_n$ for almost all n-tuples $(\alpha_1, \ldots, \alpha_n)$.*

Here, as in Theorem 1, the part of the theorem involving convergence of the integral (39) is an easy consequence of the Borel-Cantelli lemma (Lemma 1).

Using the arguments given in the preceding section, we can obtain the following result generalizing Theorem 2, from which Theorem 8 can be deduced in the same way as Theorem 1 is deduced from Theorem 2:

Theorem 9. *Let $f(q)$ $(q = 1, 2, \ldots)$ be an arbitrary sequence of nonnegative real numbers less than $1/2$ such that the series*

$$\sum_{q=1}^{\infty} f^n(q) \tag{40}$$

diverges, and suppose there exists an infinite set of positive integers Q such that

$$\sum_{q\leqslant Q} f^n(q) < c_7 \sum_{q\leqslant Q} f^n(q)\left(\frac{\varphi(q)}{q}\right)^n, \tag{41}$$

where c_7 is a constant. Then for almost all n-tuples of real numbers $\alpha_1, \ldots, \alpha_n$ there exists an infinite number of solutions of the system of inequalities (38) *in integers $q > 0$, $p_1, \ldots, p_n$ satisfying the condition $(q, p_1) = (q, p_2) = \ldots = (q, p_n) = 1$.*

To prove Theorem 9 we introduce the sets

$$\mathbf{A}_q = A_q^{(1)} \times \ldots \times A_q^{(n)},$$

where $A_q^{(i)}$ is the set of those α_i in the interval $[0, 1)$ for which the inequality

$$|\alpha_i q - p_i| < f(q), \qquad (p_i, q) = 1,$$

holds for given q and some p. Then, by (31),

$$|\mathbf{A}_q| = |A_q^{(1)}| \dots |A_q^{(n)}| = \left(2f(q)\frac{\varphi(q)}{q}\right)^n.$$

Therefore

$$\sum_{q \leqslant Q} |\mathbf{A}_q| = 2^n \sum_{q \leqslant Q} f^n(q)\left(\frac{\varphi(q)}{q}\right)^n. \tag{42}$$

From (36) we obtain

$$|\mathbf{A}_{q_1} \cap \mathbf{A}_q| = |A_{q_1}^{(1)} \cap A_q^{(1)}| \dots |A_{q_1}^{(n)} \cap A_q^{(n)}| \leqslant (16f(q_1)f(q))^n$$

for $q_1 < q$. Hence

$$\sum_{q_1,\, q \leqslant Q} |\mathbf{A}_{q_1} \cap \mathbf{A}_q| < 2(16)^n \left(\sum_{q \leqslant Q} f^n(q)\right)^2$$

for sufficiently large Q, and then the condition (41) together with (42) gives

$$\sum_{q_1,\, q \leqslant Q} |\mathbf{A}_{q_1} \cap \mathbf{A}_q| < 2 \cdot 4^n c_7^2 \left(\sum_{q \leqslant Q} |\mathbf{A}_q|\right)^2.$$

By Lemma 5, the measure of the set $\mathbf{A}$ of points falling in an infinite number of the sets $\mathbf{A}_q$ is not less than $(2 \cdot 4^n c_7^2)^{-1}$. The analogue of Theorem 7 now shows that $\mathbf{A}$ contains almost all numbers.

To deduce Theorem 8 from Theorem 9, we need only verify that

$$\sum_{q \leqslant Q} \left(\frac{\varphi(q)}{q}\right)^n > \delta Q$$

for all Q with some δ that does not depend on Q. This is an easy consequence of Lemma 4 if we choose the q_k to be all the positive

integers. In fact,

$$\sum_{k \leqslant N} \sum_{p \mid k} \frac{1}{p} \leqslant \sum_{p \leqslant N} \frac{1}{p}\frac{N}{p} < c_8 N,$$

so that the condition (16) holds. In particular, we obtain

Theorem 10. *Let $f(q)$ $(q = 1, 2, \dots)$ be an arbitrary sequence of nonincreasing nonnegative real numbers for which the series* (40) *diverges. Then for almost all n-tuples of real numbers $\alpha_1, \dots, \alpha_n$ there exists an infinite number of solutions of the inequalities* (38) *in integers $q > 0, p_1, \dots, p_n$ satisfying the condition $(q, p_1) = \dots = (q, p_n) = 1$.*

By analogy with §2, we can consider systems of inequalities

$$\max(|q_k\alpha_1 - p_1|, \dots, |q_k\alpha_n - p_n|) < f(q_k),$$

where q_k runs over some sequence of positive integers, with $(q_k, p_1) = \dots = (q_k, p_n) = 1$. Using Lemma 4, we can easily obtain the analogue of Theorem 5. As an example, we consider the case $q_k = P(k)$, where $P(x)$ is a polynomial with integer coefficients whose leading coefficient is positive, in order to obtain the analogue of Theorem 4. It is sufficient to show that

$$\sum_{k \leqslant N} \left(\frac{\varphi(P(k))}{P(k)}\right)^n > \delta_1 N \tag{43}$$

for all N and some $\delta_1 > 0$ that does not depend on N. To do this, we apply Lemma 4, obtaining

$$\sum_{k \leqslant N} \sum_{p \mid P(k)} \frac{1}{p} = \sum_{p \leqslant P(N)} \frac{1}{p} \mathscr{N}\{k \leqslant N;\ P(k) \equiv 0 \pmod{p}\}.$$

If $P(x) = P_1^{u_1} \dots P_t^{u_t}$, where the P_i are distinct irreducible polynomials with integer coefficients, then divisibility of $P(k)$ by a prime number p is equivalent to divisibility of $P_0(k) = P_1(k) \dots P_t(k)$ by

p. The discriminant of $P_0(x)$ is nonzero, and if p is different from the finite number of divisors of $P_0(x)$, then

$$\mathcal{N}\{k \leqslant N;\ P(k) \equiv 0 \pmod{p}\} \leqslant \frac{N}{p} + 1.$$

Therefore

$$\sum_{k \leqslant N} \sum_{p \mid P(k)} \frac{1}{p} \leqslant N \sum_{p} \frac{1}{p^2} + {\sum_{p \leqslant P(N)}}' \frac{1}{p} \leqslant N + \ln \ln N,$$

so that the inequality (16), and (43) as well, is satisfied.

Theorem 11. *Let λ_k $(k = 1, 2, \ldots)$ be an arbitrary nonincreasing sequence of nonnegative numbers, and let $P(x)$ be a polynomial with integer coefficients whose leading coefficient is positive. If the series*

$$\sum_{k=1}^{\infty} \lambda_k^n$$

diverges, then for almost all n-tuples of real numbers $\alpha_1, \ldots, \alpha_n$ the system of inequalities

$$\max(|\alpha_1 P(k) - a_1|, \ldots, |\alpha_n P(k) - a_n|) < \lambda_k$$

has an infinite number of solutions in integers $k, a_1, \ldots, a_n$ satisfying the condition

$$(P(k), a_1) = \ldots = (P(k), a_n) = 1.$$

As noted by Gallagher [22], unlike Theorem 1, the condition of monotonicity of $xf^n(x)$ is not necessary in Theorem 8, even if we assume that $(q, p_1, \ldots, p_n) = 1$. However, it is easy to see that in Theorem 10 the condition of monotonicity of $f(q)$ is necessary, since the existence of infinitely many solutions depends more fundamentally on the divergence of the series

$$\sum_{q=1}^{\infty} f^n(q) \left(\frac{\varphi(q)}{q}\right)^n$$

than on the divergence of the series (40).

§5. Systems of Linear Forms

We now take the next step, leading to problems of the most general kind. The first very general theorem "of the Khintchine type" was proved by A. V. Groshev [23].

Theorem 12. *Given arbitrary integers $m \geqslant 1$, $n \geqslant 1$, let $f(x)$ be a positive continuous function, defined for $x > c$, such that $x^{n-1} f^m(x)$ is a monotonically decreasing function, with $x^n f^m(x) \to 0$ $(x \to \infty)$. Then the system of inequalities*

$$|\omega_{i1} a_1 + \ldots + \omega_{in} a_n - b_i| < f(a), \quad a = \max |a_i| \quad (1 \leqslant i \leqslant m) \tag{44}$$

has infinitely many solutions in integers $a_1, \ldots, a_n, b_1, \ldots, b_m$ for almost all points

$$\omega = (\omega_{ij}) \quad (i = 1, \ldots, m, \quad j = 1, \ldots, n)$$

of mn-dimensional Euclidean space, provided that the integral

$$\int_c^{\infty} x^{n-1} f^m(x)\, dx \tag{45}$$

diverges. On the other hand, if the integral (45) converges, the system of inequalities (44) has no more than finitely many solutions in integers $a_1, \ldots, a_n, b_1, \ldots, b_m$.

We will deduce this theorem for $n \geqslant 2$ as a consequence of Theorem 15 below, dropping the condition of monotonicity of $x^{n-1} f^m(x)$ if $n \geqslant 3$ and replacing the integral (45) by the sum

$$\sum_{q=1}^{\infty} q^{n-1} f^m(q). \tag{46}$$

Given positive integers m and n, let Ω_{mn} be the space of $m \times n$ matrices ω whose components lie in the unit interval: $0 \leqslant \omega_{ij} < 1$ $(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n)$. In the space Ω_{mn} we introduce Lebesgue measure μ_{mn}, mapping Ω_{mn} in a one-to-one fashion onto the mn-dimensional unit cube $\mathbf{E}^{mn}$. Let $\mathbf{A}_{mn}$ denote the class of μ_{mn}-measurable sets over Ω_{mn}. This gives a probability space $(\Omega_{mn}, \mathbf{A}_{mn}, \mu_{mn})$.

Let us now agree on the following notation: If A is a subset of $\mathbf{E}^m$ and $\bar{x}$ is a point in $\mathbf{R}^m$, then $\bar{x} \in A \pmod 1$ means that there exists an integer vector $\bar{a} \in \mathbf{R}^m$ for which $\bar{x} + \bar{a} \in A$.† This notation is very useful for formulating the general results given in this section.

Given an integer vector $\bar{a} \neq (0)$ in $\mathbf{R}^n$, let $T = T_{\bar{a}}$ be the mapping of the space Ω_{mn} onto Ω_{1m} defined by

$$T : \omega \rightarrow (\bar{a}\omega_1, \ \ldots, \ \bar{a}\omega_m) \pmod 1, \tag{47}$$

where the ω_i are the row vectors of the matrix ω, $\bar{a}\omega_i$ is the scalar product of the vector $\bar{a}$ and ω_i, and mod 1 means that the components of the vector (47) are taken modulo 1. For $A \in \mathbf{A}_{1m}$, let $T^{-1}A$ be the preimage of A in Ω_{mn}, i.e., the set of all $\omega \in \Omega_{mn}$ which go into A under the mapping T.

The mappings (47) have two important properties. In the first place, they preserve measure:

$$|T^{-1}A| = |A|. \tag{48}$$

Here $|T^{-1}A| = \mu_{mn}(T^{-1}A)$, $|A| = \mu_{1m}(A)$, i.e., the measures are taken in the spaces where the corresponding sets are defined. Second, the sets $T^{-1}A$ are stochastically independent for linearly independent

†By an "integer vector" is meant, of course, a vector all of whose components are integers. (Translator)

integer vectors $\bar{a}_1, \bar{a}_2$, that is, if $A_1, A_2 \in \mathbf{A}_{1m}$, $T_1 = T_{\bar{a}_1}$, $T_2 = T_{\bar{a}_2}$, then

$$|T_1^{-1}A_1 \cap T_2^{-1}A_2| = |A_1|\,|A_2|. \tag{49}$$

We now prove formulas (48) and (49).

Lemma 8. *Formula* (48) *holds for every set* $A \in \mathbf{A}_{1m}$ *and every integer vector* $\bar{a} \neq (0)$.

PROOF. Let $\chi = \chi(x_1, \ \ldots, \ x_m)$ be the characteristic function of the set A, continued periodically onto $\mathbf{R}^m$ with period 1 in every variable. Expanding $\chi(\bar{x})$ in a Fourier series, we get

$$\chi(\bar{x}) \sim \alpha_0 + \sum_{\bar{\nu} \neq (0)} \alpha_{\nu} e^{2\pi i \nu x}.$$

Since $|T^{-1}A| = \int_{\mathbf{E}^{mn}} \chi(\bar{a}\omega_1, \ \ldots, \ \bar{a}\omega_m)\, d\mu_{mn}$, we find that

$$|T^{-1}A| = \alpha_0 + \sum_{\bar{\nu} \neq (0)} \alpha_{\bar{\nu}} \int_{\mathbf{E}^{mn}} \exp 2\pi i (\bar{a}\omega_1\nu_1 + \ldots + \bar{a}\omega_m\nu_m)\, d\mu_{mn}.$$

We have

$$\bar{a}\omega_1\nu_1 + \ldots + \bar{a}\omega_m\nu_m = \sum_{i=1}^{m} \sum_{j=1}^{n} \nu_i a_j \omega_{ij},$$

and hence, since $\bar{a} \neq (0)$, $\bar{\nu} \neq (0)$, we can find a pair of indices i, j such that $\nu_i \neq 0$, $a_j \neq 0$. Bearing in mind that

$$\mu_{mn} = \mu \times \ldots \times \mu, \qquad \mathbf{E}^{mn} = \mathbf{E} \times \ldots \times \mathbf{E} \tag{50}$$

are the direct products of mn one-dimensional components, we see that all the integrals vanish. Therefore

$$|T^{-1}A| = \alpha_0 = \int_{\mathbf{E}^m} \chi\, d\mu_m = |A|,$$

thereby proving formula (48).

Lemma 9. *Formula* (49) *holds for arbitrary sets* A_1, A_2 *in* $\mathbf{A}_{1m}$ *and arbitrary linearly independent integer vectors* $\bar{a}_1, \bar{a}_2 \in \mathbf{R}^n$.

The *proof* is similar to the one just given. Let $\chi^{(1)}$, $\chi^{(2)}$ be the characteristic functions of the sets A_1, A_2, respectively, continued with period 1 onto $\mathbf{R}^m$. Then

$$|T_1^{-1}A_1 \cap T_2^{-1}A_2| = \int_{\mathbf{E}^m} \chi^{(1)}(\bar{a}\omega_1, \ldots, \bar{a}\omega_m)\,\chi^{(2)}(\bar{a}\omega_1, \ldots, \bar{a}\omega_m)\, d\mu_{mn}.$$

Expanding $\chi^{(1)}$, $\chi^{(2)}$ in Fourier series

$$\chi^{(1)} \sim \alpha_0^{(1)} + \sum_{\bar{\nu}\neq(0)} \alpha_{\bar{\nu}}^{(1)} e^{2\pi i\bar{\nu}\bar{x}},$$

$$\chi^{(2)} \sim \alpha_0^{(2)} + \sum_{\bar{\lambda}\neq(0)} \alpha_{\bar{\lambda}}^{(2)} e^{2\pi i\bar{\lambda}\bar{x}},$$

we get

$$|T_1^{-1}A_1 \cap T_2^{-1}A_2| = \alpha_0^{(1)}\alpha_0^{(2)} + \ldots,$$

where the dots indicate terms of the form

$$\alpha_{\bar{\nu}}^{(1)}\alpha_{\bar{\lambda}}^{(2)} \int_{\mathbf{E}^{mn}} \exp 2\pi i \sum_{k=1}^{m} (\bar{a}_1\omega_k\nu_k + \bar{a}_2\omega_k\lambda_k)\, d\mu_{mn}, \tag{51}$$

with either $\bar{\nu} \neq (0)$ or $\bar{\lambda} \neq (0)$. The sum behind the integral sign is

$$\sum_{k=1}^{m} \omega_k (\bar{a}_1\nu_k + \bar{a}_2\lambda_k). \tag{52}$$

Since at least one of the vectors ν, $\bar{\lambda}$ is nonzero, there is an index k such that the numbers ν_k and λ_k are not both zero. Since the vectors $\bar{a}_1$, $\bar{a}_2$ are assumed to be linearly independent, we have $\bar{a}_1\nu_k + \bar{a}_2\lambda_k \neq (0)$. Hence the sum (52) contains a term $b_{ij}\omega_{ij}$, where

b_{ij} is a nonzero integer. The expansion (50) of the measure μ_{mn} and the domain of integration $\mathbf{E}^{mn}$ into one-dimensional components shows that all the integrals (51) vanish. Since $\alpha_0^{(1)} = |A_1|$, $\alpha_0^{(2)} = |A_2|$, we then get equation (49).

We can now prove the following general theorem:

Theorem 13. *Given an infinite set S of integer vectors $\bar{a} \in \mathbf{R}^n$, $\bar{a} \neq (0)$, let a set $A(\bar{a}) \in \mathbf{A}_{1m}$ be associated with each $\bar{a} \in S$, and suppose the series*

$$\sum_{\bar{a}\in S} |A(\bar{a})| \tag{53}$$

converges. Then the conditions

$$(\bar{a}\omega_1, \ldots, \bar{a}\omega_m) \in A(\bar{a}) \pmod 1, \quad \bar{a} \in S \tag{54}$$

hold only a finite number of times for almost all $\omega \in \Omega_{mn}$. However, if the series (53) diverges and every pair of vectors in S are linearly independent, then the conditions (54) hold infinitely often for almost all $\omega \in \Omega_{mn}$.

Proof. Let Ω'_{mn} be the set of those $\omega \in \Omega_{mn}$ for which (54) holds infinitely often. Since $T_{\bar{a}}^{-1}A(\bar{a})$ is the set of all $\omega \in \Omega_{mn}$ for which (54) holds for a given $\bar{a}$, we have

$$\Omega'_{mn} \subseteq \bigcup_{\substack{\bar{a}\in S\\ h(\bar{a})\geqslant h}} T_{\bar{a}}^{-1}A(\bar{a})$$

for arbitrary h, where $h(\bar{a})$ is the largest absolute value of the components of the vector $\bar{a}$ (its "height"). If the series (53) converges, then by (48) we obtain

$$|\Omega'_{mn}| \leqslant \lim_{h\to\infty} \sum_{\substack{\bar{a}\in S\\ h(\bar{a})\geqslant h}} |T_{\bar{a}}^{-1}A(\bar{a})| = \lim_{h\to\infty} \sum_{\substack{\bar{a}\in S\\ h(a)\geqslant h}} |A(\bar{a})| = 0.$$

To prove the part of the theorem pertaining to divergence of the series (53), we apply Lemma 5 to the sets $T_{\bar{a}}^{-1}A(\bar{a})$, $\bar{a} \in S$, numbered in order of increasing $h(\bar{a})$. Because of (48), we find that

$$\sum_{\substack{\bar{a}\in S \\ h(\bar{a})\leqslant h}} |T_{\bar{a}}^{-1}A(\bar{a})| = \sum_{\substack{\bar{a}\in S \\ h(\bar{a})\leqslant h}} |A(\bar{a})|.$$

Since, by hypothesis, every pair of vectors $\bar{a}_1$, $\bar{a}_2 \in S$ are linearly independent, (49) holds and we get

$$\sum_{\substack{\bar{a}_1,\ \bar{a}_2\in S \\ h(\bar{a}_1),\ h(\bar{a}_2)\leqslant h}} |T_{\bar{a}_1}^{-1}A(\bar{a}_1) \cap T_{\bar{a}_2}^{-1}A(\bar{a}_2)| = \sum_{\bar{a}_1\neq\bar{a}_2} + \sum_{\bar{a}} =$$

$$\sum_{\bar{a}_1\neq\bar{a}_2} |A(\bar{a}_1)|\,|A(\bar{a}_2)| + \sum_{\bar{a}} |A(\bar{a})|,$$

which as $h \to \infty$ is asymptotically equivalent to

$$\Bigl(\sum_{\substack{\bar{a}\in S \\ h(\bar{a})\leqslant h}} |A(\bar{a})|\Bigr)^2.$$

Therefore, by Lemma 5, we obtain

$$|\Omega'_{mn}| \geqslant 1,$$

thereby proving the theorem.

A number of consequences of Theorem 12 can be deduced by making a special choice of the system of vectors S and the sets $A(\bar{a})$.

Let P be the set of primitive integer vectors in $\mathbf{R}^n$, $n \geqslant 2$, i.e., vectors $\bar{a} = (a_1, \ldots, a_n)$ whose components are relatively prime, meaning that the greatest common divisor $(a_1, \ldots, a_n) = 1$. If $\bar{a}$, $\bar{b} \in P$ and $\bar{a} \neq \pm \bar{b}$, then $\bar{a}$ and $\bar{b}$ are linearly independent. In fact, if this is not true, there exist integers u, v such that $u\bar{a} + v\bar{b} = 0$, where it can be assumed that $(u, v) = 1$. Then u divides all the components of the vector $\bar{b}$, and hence $u = \pm 1$, and similarly $v = \pm 1$, $\bar{a} = \pm \bar{b}$, which is precluded. Next we divide P into classes, assigning to the class P_k^+ all $a \in P$ such that $a_1 = \ldots = a_{k-1} = 0$, $a_k > 0$, and to the class P_k^- all $a \in P$ such that $a_1 = \ldots = a_{k-1}$, $a_k > 0$. These classes contain only linearly independent vectors, and do not intersect for different k or different signs (+), (–). The total number of classes is $2n - 2$.

We now assume that a set $A(\bar{a}) \in \mathbf{A}_{1m}$ is associated with each primitive integer vector $\bar{a} \in \mathbf{R}^n$ and that the series

$$\sum_{a\in P} A(\bar{a}) \qquad (55)$$

diverges. Then the series summed over the vectors $\bar{a}$ belonging to some class P_k^+ or P_k^- diverges. Applying Theorem 13 to this class gives [70, 72]

Theorem 14. *Let $n \geqslant 2$ and suppose that a measurable set $A(\bar{a}) \in \mathbf{A}_{1m}$ is associated with each primitive integer vector $\bar{a} \in \mathbf{R}^n$. Then the conditions*

$$(\bar{a}\omega_1, \ldots, \bar{a}\omega_m) \in A(\bar{a}) \pmod 1$$

are satisfied infinitely often by primitive integer vectors $\bar{a}$ for almost all $\omega \in \Omega_{mn}$ if the series (55) *diverges, and only a finite number of times if the series* (55) *converges.*

Let $f(q)$ $(q = 1, 2, \ldots)$ be a sequence of real numbers in the interval $[0, 1)$, let I_q be a subinterval of $[0, 1)$ of length $f(q)$, and let $A(\bar{a}) = I_q \times \ldots \times I_q$, where $\bar{a}$ is an integer vector of height q. Then

$$\sum_{\bar{a}\in P} |A(\bar{a})| = \sum_{1\leqslant q\leqslant\infty} f^m(q) \sum_{\substack{(a_1,\ldots,a_n)=1 \\ \max|a_i|=q}} 1.$$

Calculating the inner sum, we find that it equals

$$\sum_{\substack{a_1, \ldots, a_n \\ \max|a_i|=q}} \sum_{d \mid (a_1, \ldots, a_n)} \mu(d) = \sum_{d \mid q} \mu(d) \sum_{\substack{a_1', \ldots, a_n' \\ \max|a_i'|=q/d}} 1 =$$

$$\sum_{d \mid q} \mu(d)\,[2n(1+2qd^{-1})^{n-1} + O((qd^{-1})^{n-2})] =$$

$$2^n n q^{n-1} \sum_{d \mid q} \frac{\mu(d)}{d^{n-1}} + O\left(q^{n-2} \sum_{d \mid q} \frac{|\mu(d)|}{d^{n-2}}\right). \quad (56)$$

If $n = 2$, the last expression is of the form

$$8\varphi(q) + O(2^{\nu(q)}),$$

and divergence of the series (55) is equivalent to divergence of the series

$$\sum_{q=1}^{\infty} \varphi(q) f^m(q). \quad (57)$$

However, if $n \geqslant 3$, then (56) is of the form

$$2^n n q^{n-1} \prod_{p \mid q} \left(1 - \frac{1}{p^{n-1}}\right) + O(q^{n-2} 2^{\nu(q)}),$$

where the product is taken over the prime divisors of q. This product is less than 1 and no less than

$$\prod_p \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

Therefore, if $n \geqslant 3$, divergence of the series (55) is equivalent to divergence of the series (46). Thus we obtain a result containing Theorem 12.

Theorem 15. *Let $f(q) = |I_q|$, where I_q $(q = 1, 2, \ldots)$ is an arbitrary sequence of subintervals of the interval* $[0, 1)$, *and let m, n $\geqslant 2$ be integers. Then the system of relations*

$$\omega_{i1} a_1 + \ldots + \omega_{in} a_n \in I_q \ (\mathrm{mod}\ 1) \qquad (1 \leqslant i \leqslant m) \quad (58)$$

is satisfied infinitely often by primitive integer vectors $(a_1, \ldots, a_n)$ for almost all $\omega \in \Omega_{mn}$ if the series (57) diverges in the case $n = 2$, and if the series (46) diverges in the case $n \geqslant 2$. However, the convergence of these series implies the existence of only a finite number of primitive integer vectors $(a_1, \ldots, a_k)$ satisfying (58).

In the case $n = 2$, the condition $f(q)$ or $q^{\gamma} f(q)$ be monotonic makes divergence of the series (57) equivalent to divergence of the series (46).

§6. Nonlinear Approximations

The exceptional generality of Theorems 13 and 14 allows us to deduce a number of implications involving nonlinear Diophantine approximations. This is possible because the sets $A(\bar{a})$ can be chosen to depend on the arithmetic structure of the components of the vector $\bar{a}$, with "unwanted" solutions being excluded by assigning them the empty set $A(\bar{a})$. We find ourselves in the rare situation where a sufficiently complete solution of a linear problem makes it possible to solve a number of nonlinear problems.

We now consider several examples, turning first to Theorem 14 in which it will be assumed that $m = 1$, while $n \geqslant 2$ is arbitrary. It follows from this theorem that if M is an arbitrary infinite set of primitive integer vectors $\bar{a} \in \mathbf{R}^n$ and if $\varphi(\bar{a}) \geqslant 0$ is a function defined on M such that

$$\sum_{\bar{a} \in M} \varphi(\bar{a}) = \infty,$$

then the inequality

$$\{\alpha_1 a_1 + \ldots + \alpha_n a_n\} \leqslant \varphi(\bar{a}), \qquad \bar{a} \in M$$

holds infinitely often for almost all $(\alpha_1, \ldots, \alpha_n) \in \mathbf{R}^n$ ($\{\alpha\}$ being the fractional part of α).

For M we can choose the set of all vectors of the form $(b_1^s, \dots, b_n^s)$, where $s > 0$ is a fixed integer and $b_1, \dots, b_n$ are variable integers subject to the condition that g.c.d. $(b_1, \dots, b_n) = 1$, at the same time taking $\varphi(\bar{a})$ to be b^{-n}, where

$$b = \max(|b_1|, \dots, |b_n|) \neq 0.$$

Then the series

$$\sum_{(b_1, \dots, b_n)=1} b^{-n}$$

diverges, and we see that the Diophantine inequality

$$\{\alpha_1 b_1^s + \dots + \alpha_n b_n^s\} \leqslant b^{-n}, \qquad b = \max |b_i|, \tag{59}$$

has infinitely many solutions in integers $b_1, \dots, b_n$ subject to the condition $(b_1, \dots, b_n) = 1$ for almost all $(\alpha_1, \dots, \alpha_n) \in \mathbf{R}^n$. Now let

$$\varphi(\bar{a}) = (b_1' \dots b_n'), \qquad b_i' = \max(1, |b_i|),$$

for $\bar{a} = (b_1^s, \dots, b_n^s) \in M$. Then the series

$$\sum_{(b_1, \dots, b_n)=1} (b_1' \dots b_n')^{-1}$$

diverges, and hence the number of solutions of the inequality

$$\{\alpha_1 b_1^s + \dots + \alpha_n b_n^s\} < (b_1' \dots b_n')^{-1}$$

in relatively prime integers $b_1, \dots, b_n$ is infinite for almost all $(\alpha_1, \dots, \alpha_n) \in \mathbf{R}^n$. A similar result is valid for the exponential inequalities

$$\{\alpha_1 g_1^l + \dots + \alpha_n g_n^l\} < l^{-1}, \tag{60}$$

where $g_1, \dots, g_n, l$ are positive integers with $(g_1, \dots, g_n) = 1$.

In the same way, we can obtain an unlimited number of similar examples. We now consider a general problem involving Diophantine approximation by polynomials in several variables [70, 74].

Let $\mathfrak{P}_{s,d}$ be the class of polynomials

$$P = P(x_1, \dots, x_s) = \sum_{\substack{i_1 + \dots + i_s \leqslant d \\ (i_1, \dots, i_s) \neq (0)}} \alpha_{i_1 \dots i_s} x_1^{i_1} \dots x_s^{i_s} \tag{61}$$

of degree $d \geqslant 2$ in $s \geqslant 1$ variables, with real coefficients and no constant term. On the set $\mathfrak{P}_{s,d}$ we introduce a measure, defined as Lebesgue measure in the space $\mathbf{R}^D$ of the coefficients $(\alpha_{i_1 \dots i_s})$ of these polynomials, where D is the number of all possible indices in (61), i.e., the number of solutions in integers $i_1 \geqslant 0, \dots, i_s \geqslant 0$ of the inequalities

$$i_1 + \dots + i_s \leqslant d \qquad (i_1, \dots, i_s) \neq (0). \tag{62}$$

For clarity we will assume that the coordinates of the points of $\mathbf{R}^D$ are numbered by s indices satisfying the condition (62).

Theorem 16. *Let $s \geqslant 1$, $d \geqslant 2$ be integers, and let $\lambda(q_1, \dots, q_s)$ be a nonnegative real function defined at all integer points of $\mathbf{R}^s$ and taking values $\leqslant 1$. Then for almost all polynomials $P \in \mathfrak{P}_{s,d}$ there exist infinitely many solutions of the inequality*

$$\{P(q_1, \dots, q_s)\} \leqslant \lambda(q_1, \dots, q_s) \tag{63}$$

in integers $q_1, \dots, q_s$ if the series

$$\sum_{q_1, \dots, q_s} \lambda(q_1, \dots, q_s)$$

diverges, and only finitely many solutions if the series converges.

PROOF. The theorem is a simple consequence of Theorem 13. In fact, given any integer vector $\bar{a} \in \mathbf{R}^D$, $\bar{a} \neq (0)$, we define the set $A(\bar{a})$ in accordance with the following rule: If there exists a set of integers $q_1, \dots, q_s$ such that every component $a_{i_1 \dots i_s}$ of the vector $\bar{a}$ is of the form

$$a_{l_1 \ldots l_s} = q_1^{l_1} \ldots q_s^{l_s}, \tag{64}$$

then $A(\bar{a})$ is the interval $[0, \lambda(q_1, \ldots, q_s))$, while if there is no choice of the integers $q_1, \ldots, q_s$ such that all the components of the vector $\bar{a}$ can be expressed in the form (64), then $A(\bar{a})$ is the empty set.

It is obvious that for a given $\bar{a} \in \mathbf{R}^D$ there can only exist one set of numbers $q_1, \ldots, q_n$ satisfying the condition (64), since the numbers q_i are defined by the formula $q_i = a_{0\ldots1\ldots0}$, where 1 stands in the ith position from the left. Moreover, the vectors $\bar{a}, \bar{a} \in \mathbf{R}^D$, corresponding to two sets of numbers $q_1, \ldots, q_n$ and $p_1, \ldots, p_n$ are linearly independent if the sets are distinct, since their matrix contains a submatrix

$$\begin{pmatrix} q_i & q_i^2 \\ p_i & p_i^2 \end{pmatrix}$$

of rank 2 $(q_i \neq p_i)$. Therefore, if we choose the system S in Theorem 13 to be the set of integer vectors in $\mathbf{R}^D$ satisfying the condition (64), we obtain the assertion of Theorem 16.

It is actually possible to consider systems of inequalities of the form (63) for several polynomials P simultaneously.

Choosing $s = 1$, $\lambda(q) = q^{-1}$ in Theorem 16, we find that for almost all $(\alpha_1, \ldots, \alpha_d) \in \mathbf{R}^d$ there exist infinitely many solutions of the inequality

$$\{\alpha_1 q + \alpha_2 q^2 + \ldots + \alpha_d q^d\} < q^{-1} \tag{65}$$

in integers $q > 1$. This assertion can be strengthened by regarding (65) as a relation of the form

$$\{\alpha_r q^r + \alpha_s q^s + \beta_q\} < q^{-1}, \quad r \neq s,$$

and hence as a relation

$$\alpha_r q^r + \alpha_s q^s \in J_q \pmod 1,$$

where J_q is an interval or pair of intervals of total length q^{-1}. Therefore application of Theorem 13 shows that (65) has infinitely many solutions for almost all $(\alpha_r, \alpha_s) \in \mathbf{R}^2$ with the rest of the α_i being arbitrary.

§7. Asymptotic Behavior of the Number of Solutions

Many of the theorems proved above have refinements giving the asymptotic behavior of the number of solutions of the Diophantine inequalities in question. The usual method of proving such refinements is the familiar variance method of probability theory. The following lemma is often useful. It is abstracted from the works of W. Schmidt [47, 49, 52], and is based on the idea of the well-known method of Rademacher in the theory of orthogonal series.

Lemma 10. *Let* $(\Omega, \mathbf{A}, \mu)$ *be a measure space, let* $f_k(\omega)$ $(k = 1, 2, \ldots)$ *be a sequence of nonnegative* μ*-measurable functions, and let* f_k, φ_k *be sequences of real numbers such that*

$$0 \leqslant f_k \leqslant \varphi_k \leqslant 1 \qquad (k = 1, 2, \ldots). \tag{66}$$

Suppose that

$$\int_\Omega \Big(\sum_{m<k\leqslant n} f_k(\omega) - \sum_{m<k\leqslant n} f_k \Big)^2 d\mu \leqslant c_{10} \sum_{m<k\leqslant n} \varphi_k \tag{67}$$

for arbitrary integers m, n $(m < n)$. *Then*

$$\sum_{1\leqslant k\leqslant n} f_k(\omega) = \sum_{1\leqslant k\leqslant n} f_k + O(\Phi^{1/2}(n) \ln^{3/2+\varepsilon} \Phi(n)) \tag{68}$$

for almost all $\omega \in \Omega$, *where* $\varepsilon > 0$ *is arbitrary and* $\Phi(n) = \sum_{1\leqslant k\leqslant n} \varphi_k$.

PROOF. Let J denote the class of intervals $I = (m, n]$ with integer endpoints m, n $(0 \leqslant m < n)$, where two intervals $I_1 = (m_1, n_1]$, $I_2 = (m_2, n_2]$ are said to be adjacent if $n_1 = m_2$. To keep the notation simple, it is convenient to introduce the following definitions: For $I \in J$ we write

$$\varphi(I) = \sum_{k \in I} \varphi_k, \quad f(I) = \sum_{k \in I} f_k, \quad f(I, \omega) = \sum_{k \in I} f_k(\omega),$$

where the summation is taken over all integers k in the interval I. It is obvious that

$$\varphi(I_1 + I_2) = \varphi(I_1) + \varphi(I_2)$$

for arbitrary adjacent intervals I_1, I_2, and the same is true for $f(I)$, $f(I, \omega)$.

Let

$$\varphi((0, n]) = \varphi(n),$$
$$f((0, n]) = f(n), \quad f((0, n], \omega) = f(n, \omega).$$

Then it follows from the conditions of the lemma that $\varphi(n), f(n)$, $f(n, \omega)$ are nondecreasing (the last assertion holds for every fixed $\omega \in \Omega$).

Now let $u > 0$ be an integer, and let n_u be the largest n for which $\varphi(n) < u$, so that $\varphi(n_u) < u \leqslant \varphi(n_u + 1)$. Obviously n_u is nondecreasing as a function of u, and moreover $n_{u+1} \geqslant n_u + 1$, since $\varphi(n_u + 1) \leqslant \varphi(n_u) + 1 < u + 1$, because of (66). Hence the interval $(n_u, n_v]$, $0 < u < v$, is nonempty. On the class of intervals J we introduce a mapping σ: $I \to I^\sigma$, defined by the correspondence $(u, v] \to (n_u, n_v]$. Then if $I = I_1 + I_2$, where I_1, I_2 are adjacent intervals, it follows that $I^\sigma = I_1^\sigma + I_2^\sigma$, where I_1^σ, I_2^σ are also adjacent intervals, and this property can be extended by induction to finite sums of intervals.

Let $J_{r,s}$ $(0 \leqslant s \leqslant r)$ be the subset of J consisting of the intervals of the form $(i2^s, (i+1)2^s]$, $0 \leqslant i < 2^{r-s}$. We have

$$\sum_{I \in J_{r,s}} I^\sigma = (0, 2^r] = (0, n_{2^r}],$$
$$\sum_{I \in J_{r,s}} \varphi(I^\sigma) = \varphi\Big(\sum_{I \in J_{r,s}} I^\sigma\Big) = \varphi(n_{2^r}) < 2^r. \tag{69}$$

Let $J_r = \bigcup J_{r,s}$ $(0 \leqslant s \leqslant r)$. It follows from (69) that

$$\sum_{I \in J_r} \varphi(I^\sigma) = \sum_{0 \leqslant s \leqslant r} \sum_{I \in J_{r,s}} \varphi(I^\sigma) < (r+1) 2^r. \tag{70}$$

Writing

$$g(r, \omega) = \sum_{I \in J_r} (f(I^\sigma, \omega) - f(I^\sigma))^2,$$

we have

$$\int_\Omega g(r, \omega)\, d\mu \leqslant c_{10} \sum_{I \in J_r} \varphi(I^\sigma) < c_{10}(r+1) 2^r$$

because of (67), (70). Therefore

$$\mu\{\omega \in \Omega;\ g(r, \omega) \geqslant c_{10}(r+1) r^{1+\varepsilon} 2^r\} < r^{-1-\varepsilon},$$

and, by the Borel-Cantelli lemma, for almost all $\omega \in \Omega$ the inequality

$$g(r, \omega) < c_{10}(r+1) r^{1+\varepsilon} 2^r \tag{71}$$

holds for every $r \geqslant r(\omega)$.

Now let v be an arbitrary positive integer. The interval $(0, v]$ can be represented as a sum of no more than $r = [\log_2 v] + 1$ adjacent intervals I from J_r (this is the geometric interpretation of the expansion of v in the binary number system). Let $J_r(v)$ denote the set of these intervals. Correspondingly, the interval $(0, n_v]$ decomposes into a sum of no more than r intervals I^σ, $I \in J_r(v)$:

$$(0, v] = \sum_{I \in J_r(v)} I \Rightarrow (0, n_v] = \sum_{I \in J_r(v)} I^\sigma.$$

Using the Cauchy-Schwarz inequality, we deduce from the formula

$$f(n_v, \omega) - f(n_v) = \sum_{I \in J_r(v)} (f(I^\sigma, \omega) - f(I^\sigma))$$

that

$$(f(n_v, \omega) - f(n_v))^2 \leqslant r \sum_{I \in J_r(v)} (f(I^\sigma, \omega) - f(I^\sigma))^2,$$

which does not exceed $rg(r, \omega)$. It now follows from (71) that

$$f(n_v, \omega) = f(n_v) + O(2^{r/2} r^{(3/2)+\varepsilon})$$

for almost all $\omega \in \Omega$. The remainder is of the form

$$O(v^{1/2} \ln^{(3/2)+\varepsilon} v) = O(\varphi^{1/2}(n_v) \ln^{(3/2)+\varepsilon} \varphi(n_v)),$$

which proves the lemma for $n = n_v$. But $f(I, \omega)$ is nondecreasing, since $f(n, \omega) \geqslant 0$. Therefore $n_v \leqslant n < n_{v+1}$ implies

$$f(n_v, \omega) \leqslant f(n, \omega) \leqslant f(n_{v+1}, \omega).$$

Moreover, from (66) and the definition of n_v we get $\varphi(n_v) = v + O(1)$, so that

$$f(n_{v+1}) - f(n_v) \leqslant \varphi(n_{v+1}) - \varphi(n_v) = O(1),$$

which gives (68) in the general case.

Using Lemma 10 and formulas (48) and (49), we can easily prove

Theorem 17. *Under the conditions of Theorem* 13, *let* $N(\omega, h)$ *be the number of vectors* $\bar{a} \in S$ *of height* $h(\bar{a}) \leqslant h$ *satisfying* (54) *for a given* $\omega \in \Omega_{mn}$, *and suppose the series* (54) *diverges. Then*

$$N(\omega, h) = \Phi(h) + O(\Phi^{1/2}(h) \ln^{(3/2)+\varepsilon} \Phi(h))$$

for almost all $\omega \in \Omega_{mn}$, *where* $\varepsilon > 0$ *is arbitrary, and*

$$\Phi(h) = \sum_{\substack{\bar{a} \in S \\ h(\bar{a}) \leqslant h}} |A(\bar{a})| .$$

PROOF. In fact, let $\nu(\bar{a})$, $\bar{a} \in S$, be an arbitrary enumeration of the vectors $\bar{a} \in S$. We introduce the characteristic functions $\chi_\nu(\omega)$ of the sets $T_\nu^{-1} A(\bar{a}_\nu)$, $\nu = \nu(\bar{a})$, and we set

$$f_\nu = \varphi_\nu = |A(\bar{a}_\nu)|.$$

Then (48) implies

$$\int_{\Omega_{mn}} \chi_\nu(\omega)\, d\mu_{mn} = |T_\nu^{-1} A(\bar{a}_\nu)| \neq |A(\bar{a}_\nu)|,$$

$$\int_{\Omega_{mn}} (\chi_\nu(\omega) - f_\nu)^2\, d\mu_{mn} = f_\nu - f_\nu^2 \leqslant f_\nu.$$

Similarly, it follows from (49) that

$$\int_{\Omega_{mn}} (\chi_\nu(\omega) - f_\nu)(\chi_{\nu'}(\omega) - f_{\nu'})\, d\mu_{mn} = \int_{\Omega_{mn}} \chi_\nu(\omega) \chi_{\nu'}(\omega)\, d\mu_{mn} - f_\nu f_{\nu'} = |T_\nu^{-1} A(\bar{a}_\nu) \cap T_{\nu'}^{-1} A(\bar{a}_{\nu'})| - f_\nu f_{\nu'} = 0$$

if $\nu \neq \nu'$. Therefore we get

$$\int_{\Omega_{mn}} \Big(\sum_{u<\nu\leqslant v} \chi_\nu(\omega) - \sum_{u<\nu\leqslant v} f_\nu \Big)^2 d\mu_{mn} = \sum_{u<\nu\leqslant v} \int_{\Omega_{mn}} (\chi_\nu(\omega) - f_\nu)^2\, d\mu_{mn} < \sum_{u<\nu\leqslant v} \varphi_\nu$$

for arbitrary integers u, v $(u < v)$, which corresponds to the condition (67) of Lemma 10. Thus

$$\sum_{\nu\leqslant v} \chi_\nu(\omega) = \Phi_\nu(v) + O\left(\Phi_\nu^{1/2}(v)\ln^{(3/2)+\varepsilon}\Phi_\nu(v)\right) \tag{72}$$

for almost all $\omega \in \Omega_{mn}$, where

$$\Phi_\nu(v) = \sum_{\substack{\nu(\bar{a})\leqslant v\\ \bar{a}\in S}} |A(\bar{a})|.$$

In particular, if the enumeration $\nu(\bar{a})$ is such that the vectors $\bar{a} \in S$ are numbered in the order of increasing height $h(\bar{a})$, i.e., if for every h the set of all $\bar{a} \in S$ for which

$$\nu(\bar{a}) \leqslant \mathscr{N}\{\bar{a};\ h(\bar{a}) \leqslant h,\ \bar{a} \in S\}$$

coincides with the set $\{\bar{a};\ h(\bar{a}) \leqslant h,\ \bar{a} \in S\}$, then we deduce the assertion of the theorem from (72).

The asymptotic behavior in Theorem 14 is established in exactly the same way.

By this method we can easily find the asymptotic behavior of the number of solutions of specific inequalities and systems of inequalities, for example, of the form (58)-(60), (63), (65). In particular, in the last case we get

$$N(a;\ \alpha_1, \ldots, \alpha_d) = \ln Q + O(\ln^{1/2} Q \ln\ln^{(3/2)+\varepsilon} Q)$$

for the number of solutions of the inequality (65) in positive integers $q \leqslant Q$ for almost all pairs α_r, α_s $(r \neq s)$ and arbitrary fixed α_i $(1 \leqslant i \leqslant d;\ i \neq r, s)$.

However, Theorem 16 does not include many of the Diophantine inequalities considered earlier, namely those pertaining to rational approximations to one or several numbers (§1-4), in which case the solutions are one-dimensional vectors, i.e., integers q, and hence are always linearly dependent. The latter fact precludes the validity of equation (49), which is the basis for the deduction of Theorem 17 from Lemma 10. Nevertheless, Lemma 10 is useful in many cases, if we can establish a coarser version of (49) that is "not too crude."

Let us consider a simple problem in which the above complication arises. Let I_q $(q = 1, 2, \ldots)$ be a sequence of intervals inside the unit interval $[0, 1)$. We will look for restrictions that must be imposed on the lengths of the intervals $|I_q|$ in order to make the number $N(q; \bar{\alpha})$ of times that the conditions

$$\alpha_q \in I_q \pmod 1, \quad q \leqslant Q, \tag{73}$$

hold for almost all α satisfy the asymptotic formula

$$N(Q;\ \alpha) \sim \sum_{q\leqslant Q} |I_q| \qquad (Q \to \infty) \tag{74}$$

if the series

$$\sum_{q=1}^{\infty} |I_q| \tag{75}$$

diverges. We know from what has already been said that even in the case where the intervals I_q are of the form $(0, f(q))$, divergence of the series (75) without extra conditions (like the monotonicity of $f(q)$) does not allow us to assert that the formulas (73) hold infinitely often on a set of positive measure. Nevertheless, we can prove the following

Theorem 18. *For almost all* α

$$N(Q;\ \alpha) = \Phi(Q) + O(\Psi^{1/2}(Q)\ln^{(3/2)+\varepsilon}\Psi(Q)),$$

where $\Phi(Q) = \sum_{q\leqslant Q} |I_q|$, $\Psi(Q) = \sum_{q\leqslant Q} |I_q|\,\tau(q)$, $\tau(q)$ *is the number of divisors of* q, *and* $\varepsilon > 0$, *is arbitrary.*

Since $\tau(q) \ll q^\varepsilon$, we see that if the inequality $\Phi(Q) \gg Q^\delta$ holds for all $Q > 0$ with some $\delta > 0$, then the asymptotic formula (74) holds. However, if the sequence $|I_q|$ decreases monotonically, then, bearing in mind that

$$\sum_{p\leqslant q} \tau(p) \ll q \ln q$$

for all q and making an Abel transformation in the expression for $\Psi(Q)$, we obtain

$$\Psi(Q) \ll \Phi(Q) \ln Q.$$

In this case the asymptotic formula (74) is true for $\Phi(Q) \gg (\ln Q)^{1+\delta}$.

To prove Theorem 18, we need Lemma 9 and the following

Lemma 11. *Let A and B be intervals inside the unit interval* **E**, *and for positive integers q let T_q be the mapping $\omega \to q\omega \pmod 1$ of the interval* **E** *onto itself. Then*

$$|T_p^{-1}A \cap T_q^{-1}B| = |A||B| + O\left(|A| \frac{(p, q)}{p}\right) \tag{76}$$

for arbitrary positive integers p, q.

PROOF. We again use Fourier series, as in the proof of (49). Let $\chi_A(x)$ be the characteristic function of the set A, with period 1, extended onto the whole real line, and let $\chi_B(x)$ be defined similarly for B. Suppose that

$$\chi_A(x) = \sum_m a_m e^{2\pi i m x}, \quad \chi_B(x) = \sum_n b_n e^{2\pi i n x}. \tag{77}$$

Then

$$I_{p,q} = \int_0^1 \chi_A(px)\chi_B(qx)\,dx = |T_p^{-1}A \cap T_q^{-1}B|,$$

and the expansions (77) give

$$I_{p,q} = \sum_{m,n} a_m b_n \int_0^1 e^{2\pi i (mp+nq)x}\,dx = \sum_{\substack{m,n\\ mp+nq=0}} a_m b_n.$$

Let $d=(p, q)$, $p=dp'$, $q=dq'$. Then p' divides n, q' divides m, $n=p'k$, $m=q'k$, and we get

$$I_{p,q} = \sum_k a_{-q'k} b_{p'k}.$$

This formula differs greatly from the one that we had in the course of proving equation (49). Bearing in mind that

$$a_m = O\left(\frac{1}{|m|}\right), \quad b_n = O\left(\frac{1}{|n|}\right) \qquad (m \neq 0, \ n \neq 0),$$

we obtain

$$I_{p,q} = a_0 b_0 + O\left(\sum_{k=1}^{\infty} \frac{1}{p'q'k^2}\right) = |A||B| + O\left(\frac{(p, q)^2}{pq}\right).$$

In some cases this formula is stronger than (76), but it is not suitable for proving Theorem 18.

To obtain (76), we need only improve the previous argument by using smoothed functions $\chi_B^+(x)$ and $\chi_B^-(x)$ instead of $\chi_B(x)$, in accordance with the method of I. M. Vinogradov ([84], p. 23). Let $B=(\alpha, \beta)$, $0<\delta\leqslant\frac{1}{2}\min(1-(\beta-\alpha),\ \beta-\alpha)$. We write

$$\chi_B^+(x) = \begin{cases} 1, & \text{if} \quad \alpha \leqslant x \leqslant \beta, \\ \dfrac{x-\alpha}{\delta}+1, & \text{if} \quad \alpha-\delta \leqslant x \leqslant \alpha, \\ -\dfrac{x-\beta}{\delta}+1, & \text{if} \quad \beta \leqslant x \leqslant \beta+\delta, \\ 0 & \text{otherwise} \end{cases}$$

and define $\chi_B^-(x)$ similarly, replacing α by $\alpha+\delta$ and β by $\beta-\delta$. It is easy to see that the functions $\chi_B^+(x)$ and $\chi_B^-(x)$ are continuous and piecewise differentiable, with

$$\chi_B^-(x) \leqslant \chi_B(x) \leqslant \chi_B^+(x),$$

and that the Fourier coefficients b_m^+ and b_m^- of these functions satisfy the estimates

$$b_0^+ = |B| + O(\delta), \qquad b_0^- = |B| + O(\delta),$$
$$|b_m^+| + |b_m^-| \leqslant \min\left(|B|, \frac{1}{\delta m^2}\right), \qquad m \neq 0.$$

We now find that

$$I_{p,q} \leqslant \int_0^1 \chi_A(px)\chi_B^+(qx)\,dx = a_0 b_0^+ + \sum_{k\neq 0} a_{-q'k} b_{p'k} =$$
$$|A||B| + O(|A|\delta) + O(|A|)\sum_{k=1}^{\infty} \min\left(|B|, \frac{1}{\delta k^2 (p')^2}\right).$$

It can be assumed that $|B| < 1/2$, since otherwise we can decompose B into a sum of intervals of length less than 1/2 and prove (76) for them, from which we get (76) for B as well. Let $\delta = \min\left(\frac{1}{2}|B|, \frac{1}{2p'}\right)$. Then if $|B| < 1/p'$,

$$\sum_{k=1}^{\infty} \min\left(|B|, \frac{1}{\delta k^2 (p')^2}\right) \leqslant$$
$$\sum_{1\leqslant k<(|B|p')^{-1}} |B| + \sum_{k>(|B|p')^{-1}} \frac{1}{\delta k^2 (p')^2} = O\left(\frac{1}{p'}\right)$$

and hence we obtain

$$I_{p,q} \leqslant |A||B| + O(|A|/p'). \tag{78}$$

If $|B| \geqslant 1/p'$, we have the estimate $O(1/p')$ for the sum and we again get (78). Similarly, the opposite inequality is obtained if we take $\chi_B^-(x)$ instead of $\chi_B^+(x)$. In this way, we finally establish (76).

To prove Theorem 18, we introduce the characteristic functions $\chi_q(x)$ of the intervals I_q. Then

$$N(Q;\alpha) = \sum_{q\leqslant Q} \chi_q(\alpha q).$$

Because of Lemma 10, our assertion will be justified if we establish that

$$J_{m,M} =$$
$$\int_0^1 \left(\sum_{m<q\leqslant M} \chi_q(qx) - \sum_{m<q\leqslant M} I_q\right)^2 dx \ll \sum_{m<q\leqslant M} |I_q|\tau(q)$$

for arbitrary positive integers m, M $(m < M)$. Using (76), we find that

$$\int_0^1 (\chi_p(px) - |I_p|)(\chi_q(qx) - |I_q|)\,dx =$$
$$|T_p^{-1} I_p \cap T_q^{-1} I_q| - |I_q||I_p| =$$
$$O\left(\min\left(|I_p|\frac{(p,q)}{p}, |I_q|\frac{(p,q)}{q}\right)\right)$$

if $p \neq q$. Moreover,

$$\int_0^1 (\chi_p(xp) - |I_p|)^2\,dx \leqslant |I_p|.$$

Therefore we have

$$J_{m,n} \ll \sum_{m<p\leqslant M} |I_p| + \sum_{m<p<q\leqslant M} |I_q|\frac{(p,q)}{q}. \tag{79}$$

Since

$$\sum_{p<q} (p,q) \leqslant \sum_{d|q} d \sum_{p'<q/d} 1 < q\tau(q);$$

we get

$$J_{m,M} \ll \sum_{m<q\leqslant M} |I_q|\tau(q),$$

as was to be proved.

Again let I_q $(q = 1, 2, \ldots)$ be a sequence of intervals lying inside the unit interval $[0, 1)$, and consider the system of conditions

$$\alpha_i q \in I_q \ (\mathrm{mod}\ 1) \qquad (i = 1, 2, \ldots, n), \tag{80}$$

where the α_i lie in the unit interval, $n \geqslant 2$. Let $N\{Q;\ \alpha_1, \ldots, \alpha_n\}$ be the number of those $q \leqslant Q$ satisfying (80). Then Theorem 18 has the following simple generalization:

Theorem 19. *For almost all* $\alpha_1, \ldots, \alpha_n \in \mathbf{E}^n$

$$N\{Q;\ \alpha_1, \ldots, \alpha_n\} = \Phi_n(Q) + O\left(\Psi_n^{1/2}(Q)\ln^{3/2+\varepsilon}\Psi_n(Q)\right),$$

where

$$\Phi_n(Q) = \sum_{q\leqslant Q} |I_q|^n,$$

$$\Psi_n(Q) = \sum_{q\leqslant Q} |I_q|^n \gamma(q),\ \ \gamma(q) =$$
$$1 + q^{-1}\sum_{p<q} |I_p|^{n-1}(p, q) + q^{-2}\sum_{p<q} |I_p|^{n-2}(p, q)^2 < 3\tau(q).$$

The *proof* of this theorem is completely analogous to the proof of Theorem 18. Let $\mathbf{A}_q$ be the set of $(\alpha_1, \ldots, \alpha_n) \in \mathbf{E}^n$ satisfying (80). Then

$$\mathbf{A}_q = T_q^{-1}I_q \times \ldots \times T_q^{-1}I_q, \qquad |\mathbf{A}_q| = |I_q|^n.$$

By Lemma 11,

$$|\mathbf{A}_p \cap \mathbf{A}_q| = |T_p^{-1}I_p \cap T_q^{-1}I_q|^n =$$
$$\left(|I_p||I_q| + O\left(\min\left(|I_p|\frac{(p, q)}{p},\ |I_q|\frac{(p, q)}{q}\right)\right)\right)^n$$

for arbitrary integers p, q. Hence we have

$$2\sum_{Q_1\leqslant p<q\leqslant Q_2} |\mathbf{A}_p \cap \mathbf{A}_q| =$$
$$\left(\sum_{Q_1<q\leqslant Q_2} |I_q|^n\right)^2 + O\left(\sum_{Q_1<q\leqslant Q_2} |I_q|^n\right) +$$
$$O\left(\sum_{Q_1<q\leqslant Q} |I_q|^n \sum_{k=1}^{n} q^{-k} \sum_{Q_1<p<q} |I_p|^{n-k}(p, q)^k\right)$$

for arbitrary Q_1, Q_2 $(Q_1 < Q_2)$. Since

$$\sum_{p<q} |I_p|^{n-k}(p, q)^k < \sum_{d\mid q} d^k \sum_{p'\leqslant qd^{-1}} 1 = q^k \sum_{d\mid q} d^{-k+1} \ll q^k$$

for $k \geqslant 3$, we get the assertion of the theorem.

In Theorems 18 and 19 no restrictions whatsoever are imposed on the way the lengths of the intervals I_q vary with q. Therefore we can regard the conditions (73) and (80) as holding for the numbers q_k of an increasing sequence. Assuming, for example, that $q_l \mid q_k$ if $l \leqslant k$, we find that N_k, the number of times the conditions

$$\alpha_i q_k \in J_k \ (\mathrm{mod}\ 1) \qquad (i = 1, \ldots, n), \quad k \leqslant K$$

hold, satisfies the asymptotic formula

$$N_K = S_K + O\left(S_K^{1/2}\ln^{3/2+\varepsilon}S_K\right),$$

where $S_K = \sum_{k\leqslant K} |J_k|^n$, since

$$\sum_{l<k} (q_l, q_k) \ll q_k, \quad \sum_{l<k} (q_l, q_k)^2 \ll q_k^2$$

(if $n = 1$ we must use the estimate (79), rather than Theorem 18). Similarly, by introducing special restrictions on the way the intervals I_q vary with q, we can derive many other consequences. It is also clear that instead of considering the incidence of the fractional parts of $\alpha_1 q, \ldots, \alpha_n q$ in the cube $I_q \times \ldots \times I_q$, we can study their incidence in arbitrary rectangular sets.

By considering the intersection of a large number of sets of the form $T_q^{-1} A_q$, we can carefully investigate the number of times the conditions (73) or (80) hold, obtaining analogues of the law of the iterated logarithm, the central limit theorem, and so on, for various sequences of positive integers q_k $(k = 1, 2, \ldots)$. For example, for the above-mentioned sequences q_k satisfying the condition $q_l \mid q_k$ if $l \leqslant k$, we have

$$N_K = S_K + O\left(S_K^{1/2} (\ln \ln S_K)^{3/2}\right)$$

for almost all $(\alpha_1, \ldots, \alpha_n) \in \mathrm{E}^n$, where this estimate cannot be improved, and the measure of the set of $(\alpha_1, \ldots, \alpha_n) \in \mathrm{E}^n$ for which

$$t_1 T_K < N_K - S_K < t_2 T_K, \quad T_K = \sum_{k \leqslant K} |J_k|^n (1 - |J_k|)^n \to \infty$$

as $K \to \infty$ has the limit

$$\frac{1}{\sqrt{2\pi}} \int_{t_1}^{t_2} e^{-t^2/2}\, dt \qquad (-\infty \leqslant t_1 < t_2 \leqslant \infty).$$

Results of this kind are based on the fact that the characteristic functions of the sets $T_q^{-1} A_q$ are weakly dependent random variables for suitable sequences q_k.

§8. Survey of Other Results and Problems

A. Walfisz [91] was the first to consider the problem of the solutions of the inequality (1) in integers p, q, when q does not range over all the positive integers. He proved that if the integral (2) diverges and if the ratio $f(x)/f(2x)$ is bounded for $x > c$, then for almost all α the number of solutions of (1) in integers p, $q > 0$

satisfying the condition $q \not\equiv 2 \pmod 4$ is infinite. Walfisz used this theorem to study the behavior of the function

$$\sum_{n=1}^{\infty} z^{n^2} \qquad (|z| < 1)$$

near its circle of convergence.

Cassels [10] studied the one-sided inequalities

$$0 \leqslant \alpha - \frac{p}{q_k} < \frac{f(q_k)}{q_k}, \tag{81}$$

where q_k runs over a rather general sequence of positive integers. Given a sequence of positive integers q_k, let M_n denote the number of fractions of the form p/q_n $(0 < p < q_n)$ which do not coincide with any fraction of the form p'/q_m $(0 < p' < q_m)$, $m < n$. Cassels calls a sequence q_k a Σ-sequence if

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n \leqslant N}' M_n q_n^{-1} > 0.$$

He shows that although not every sequence q_k is a Σ-sequence, the most interesting sequences are of this kind, like, for example, $q_k = k^2$, $q_k = l^k$, $q_k = k!$. His basic result is the following

Theorem 20. *If $f(q)$ is monotonically decreasing and q_k is a Σ-sequence, then the inequality* (81) *has infinitely many solutions for almost all α if the series* (5) *diverges, and only a finite number of solutions if the series converges.*

We recommend that the reader compare Theorem 20 with Theorem 5. An idea of Cassels' method of reasoning can be formed by examining the contents of Chapter VII of his book [13].

Recently Erdös [19] obtained the following result:

Theorem 21. *Let $\varepsilon > 0$ and suppose the series*

$$\sum_{k=1}^{\infty} \varphi(q_k)\, q_k^{-2} \tag{82}$$

diverges, where $\varphi(q)$ is the Euler function. Then the inequality

$$\left|\alpha - \frac{p}{q_k}\right| < \frac{\varepsilon}{q_k^2}, \qquad (p, q_k) = 1,$$

has infinitely many solutions for almost all α.

Since the irreducible fraction p/q satisfying the inequality $\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$ is a convergent of the continued fraction expansion of α, we can deduce from Theorem 21 a necessary and sufficient condition for infinitely many denominators $q_n(\alpha)$ of the nth convergents of the expansion of α to belong to a given sequence q_k for almost all α. This condition is the divergence of the series (82). This strengthens the results of Hartman and Szüsz [25].

The assertion of Theorem 21 is quite close to the unproved hypothesis of Duffin and Schaeffer (see the end of §2). According to Erdös [19], the argument and technique used by him to prove Theorem 21 can probably be used to prove the hypothesis of Duffin and Schaeffer.

Following Cassels' ideas, Gallagher [21, 22] considered systems of inequalities

$$0 \leqslant \alpha_i q - p_i < f(q) \qquad (i = 1, 2, \ldots, n), \tag{83}$$

and also more general relations

$$(\alpha_1 q - p_1, \ldots, \alpha_n q - p_n) \in A_q, \tag{84}$$

where the A_q are subsets of $\mathbf{R}^n$ satisfying the following condition: If $(x_1, \ldots, x_n) \in A_q$ and $0 \leqslant x_1' \leqslant x_1, \ldots, 0 \leqslant x_n' \leqslant x_n$, then $(x_1', \ldots, x_n') \in A_q$.

Theorem 22. *Let every set A_q $(q = 1, 2, \ldots)$ have the property just indicated, and suppose the numbers $|A_q|$ decrease monotonically and the series*

$$\sum_{q=1}^{\infty} |A_q| \tag{85}$$

diverges. Then for almost all $(\alpha_1, \ldots, \alpha_n) \in \mathbf{R}^n$ the conditions (84) hold infinitely often in integers, where the numbers $q, p_1, \ldots, p_n$ are relatively prime. If the series (85) converges, then the condition (84) holds only a finite number of times for almost all $(\alpha_1, \ldots, \alpha_n) \in \mathbf{R}^n$.

From this theorem we can deduce an interesting corollary, which asserts that the inequality

$$\|\alpha_1 q\| \ldots \|\alpha_n q\| < \frac{1}{q(\ln q)^n}$$

is solvable by an infinite number of integers $q > 1$ for almost all $(\alpha_1, \ldots, \alpha_n) \in \mathbf{R}^n$. If we replace n by $n + \varepsilon$, $\varepsilon > 0$, in the right-hand side of this inequality, then the number of solutions will be finite for almost all $(\alpha_1, \ldots, \alpha_n) \in \mathbf{R}^n$. This last fact was known before Gallagher's work (see Spencer [60]).

Schmidt [49, 52] continued and developed the work of LeVeque [44] and Erdös [18], deriving the asymptotic behavior of the number of solutions of (83) for $q \leqslant Q$ and almost all $(\alpha_1, \ldots, \alpha_n)$.

Theorem 23. *Let $f_1(q), \ldots, f_n(q)$ be positive functions of a positive integer argument q, and let*

$$\psi(q) = \prod_{i=1}^{n} f_i(q), \quad \psi(Q) = \sum_{q \leqslant Q} \psi(q),$$
$$\Omega(Q) = \sum_{q \leqslant Q} \psi(q) q^{-1}.$$

Suppose that $\psi(q)$ is monotonically decreasing, and let $N(Q; \alpha_1, \ldots, \alpha_n)$ be the number of solutions of the inequalities

$$0 \leqslant \alpha_i q - p_i < f_i(q) \qquad (i = 1, \ldots, n)$$

in integers $p_1, \ldots, p_n, q \leqslant Q$. Then

$$N(Q; \alpha_1, \ldots, \alpha_n) = \psi(Q) + O(\psi^{1/2}(Q)\,\Omega^{1/2}(Q)\,\ln^{2+\varepsilon}\psi(Q))$$

for almost all $(\alpha_1, \ldots, \alpha_n) \in \mathbf{R}^n$, where $\varepsilon > 0$ is arbitrary.

Schmidt [52] obtained an analogous theorem for systems of linear forms. Moreover, he considered more general problems of the following type:

Theorem 24. *Let $P_1(q), \ldots, P_n(q)$ be nonconstant polynomials with integer coefficients, and let*

$$I_{j1} \supseteq I_{j2} \supseteq \ldots$$

be a sequence of nested intervals on the real circle of unit circumference. Let

$$\psi(q) = |I_{1q}| \ldots |I_{nq}|, \qquad \psi(Q) = \sum_{q \leq Q} \psi(q),$$

and let $N(Q; \alpha_1, \ldots, \alpha_n)$ be the number of those $q \leqslant Q$ for which

$$\alpha_i P_j(q) \in I_{jq} \pmod 1 \qquad (j = 1, \ldots, n).$$

Then, given any $\varepsilon > 0$,

$$N(Q; \alpha_1, \ldots, \alpha_n) = \psi(Q) + O(\psi(Q)^{1/2+\varepsilon})$$

for almost all $(\alpha_1, \ldots, \alpha_n) \in \mathbf{R}^n$.

We note that in Theorems 22-24 it is assumed that the quantities A_q and $\psi(q)$ decrease monotonically as q increases. Gallagher [22] obtained the asymptotic behavior of the number of solutions of (83) for $n \geqslant 2$, subject to the condition $(q, p_1, \ldots, p_n) = 1$, without assuming that $f(q)$ is a monotonic sequence.

Some generalizations and refinements of the above results of Schmidt and Gallagher have been obtained by Ennola [17].

As is clear from the foregoing, the most important unsolved problem is the Duffin and Schaeffer hypothesis. The proof of this hypothesis is probably quite complicated, since the hypothesis involves in concealed form deep facts about the uniform distribution of Farey fractions (irreducible rational fractions p/q) on the unit interval. It is well known that some facts pertaining to the uniform distribution of Farey fractions are equivalent to the Riemann hypothesis on the zeros of the zeta function [42, 82]. It is possible

that purely combinatorial arguments on the distribution of Farey fractions can be avoided, by making a more careful analysis of multiple intersections of the intervals defined by the inequality (1) (see §14 of Chapter 2).

The study of simultaneous approximations (38) subject to the condition $(q, p_1) = \ldots = (q, p_n) = 1$ is probably a problem of the same degree of complexity as in the case $n = 1$. The expected result, corresponding to the Duffin and Schaeffer hypothesis, is that there will be infinitely many solutions for almost all $(\alpha_1, \ldots, \alpha_n) \in \mathbf{R}^n$ if the series

$$\sum_{q=1}^{\infty} f^n(q) \left(\frac{\varphi(q)}{q}\right)^n$$

diverges. It can also be expected that the number of solutions of the inequalities (38) in integers $q \leqslant Q$ and integers $p_1, \ldots, p_n$ subject to the condition $(q, p_1) = \ldots = (q, p_n) = 1$ is asymptotically equal to

$$2^n \sum_{q \leq Q} f^n(q) \left(\frac{\varphi(q)}{q}\right)^n \qquad (Q \to \infty).$$

Although we have considered many problems involving nonlinear Diophantine approximations, none of these problems was nonlinear in the full sense, since the variable integer, implicit in the symbol $\{\ldots\}$, is a linear term. As of now, no metric theory of (fully) nonlinear Diophantine approximations has been constructed. The working out of such a theory is a very topical problem.