Exercise sheet – Topics in Discrepancy.
Tel Aviv University, Fall 2016

Throughout we use the following notation.

- $U^d = [0,1)^d$ and Vol is the d-dimensional Lebesgue measure restricted to U^d. We sometimes identify U^d with the d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ by the map $X \mapsto x + \mathbb{Z}^d$.
- For a collection of integrable functions $\mathcal{F}$ in a probability measure space $(X, \mathcal{B}, \mu)$, and for a sequence $(x_i)_{i=0}^{N-1}$ in X,

$$D\left((x_i)_{i=0}^{N-1}; \mathcal{F}\right) = \sup_{f \in \mathcal{F}} \left| \sum_{i=0}^{N-1} f(x_i) - \int_X f \, d\mu \right|.$$

  For a collection of subsets $\mathcal{S} \subset \mathcal{B}$, $\mathcal{F}(\mathcal{S})$ is the collection of indicators $\{\chi_S : s \in \mathcal{S}\}$ and $D((x_i); \mathcal{S}) = D((x_i); \mathcal{F}(\mathcal{S}))$. If (X, μ) are not specified and d is specified, then we take $(X, \mu) = \left(U^d, \mathrm{Vol}\right)$.
- $\mathcal{R}_d$ is the collection of axis-parallel boxes in U^d, and $\mathcal{R}_d$ is the sub-collection of elements of $\mathcal{R}_d^*$ with a corner at the origin.
- For a lattice Λ in $\mathbb{R}^d$, $\mathrm{Nm}(\Lambda) = \inf_{(v_1,\dots,v_d) \in \Lambda \smallsetminus \{0\}} |v_1 \cdots v_d|$.

1. Prove that for each d, N and each sequence $(x_i)_{i=0}^{N-1}$ in U^d,

$$D((x_i); \mathcal{R}_d^*) \leqslant D((x_i); \mathcal{R}_d) \leqslant 2^d D((x_i); \mathcal{R}_d^*).$$

2. Let $g : \mathbb{N} \to \mathbb{R}_+$ be a non-decreasing function.

(a) Suppose that for each N there is a sequence $\left(x_i^{(N)}\right)_{i=0}^{N-1} \subset U^d$ with $D\left(\left(x_i^{(N)}\right)_{i=0}^{N-1}; \mathcal{R}_d\right) = O(g(N))$. Let $P : \mathbb{R}^d \to \mathbb{R}^{d-1}$ be the projection omitting the d-th entry. For each N, assume the elements $\left(x_i^{(N)}\right)_{i=0}^{N-1}$ are ordered by increasing d-th entry, and let $y_i^{(N)} = P\left(x_i^{(N)}\right) \in \mathbb{R}^{d-1}$, $i = 0, \dots, N-1$. Show that for each N and for each $k \in \{1, \dots, N-1\}$, $D\left(\left(y_i^{(N)}\right)_{i=0}^{k}; \mathcal{R}_{d-1}\right) = O(g(k))$.

(b) By combining the sequences constructed in (a) show (under the same assumption) that there is a sequence $(y_i)_{i=0}^{\infty} \subset U^{d-1}$ such that for any N, $D_N\left((y_i)_{i=0}^{N-1}; \mathcal{R}_{d-1}\right) = O(g(N))$.

3. Suppose X is a compact metric space and μ is a Borel probability measure on X. Suppose that $(x_i)_{i=0}^{\infty}$ is a sequence in X such that for all continuous functions $f : X \to \mathbb{R}$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(x_i) = \int_X f \, d\mu. \tag{1}$$

Show that (1) also holds for functions f such that

$$\mu\left(\{x \in X : f \text{ is not continuous at } x\}\right) = 0.$$

4. *'Impossibility of a just distribution' (van Aardenne-Ehrenfest):* Show that for any sequence $(x_i)_{i=0}^{\infty}$ in U^1, the function which sends $N \in \mathbb{N}$ to

$$\sup\{\#(A \cap (x_i)_{i=0}^{N-1}) - \#(B \cap (x_i)_{i=0}^{N-1}) : A \text{ and } B \text{ are intervals of equal length in } U^1\}$$

is unbounded.

5. Let $(x_n)_{n=0}^{N-1}$ be N points in U^1 and let k so that $\sum_{h=1}^{k} \left|\sum_{n=0}^{N-1} e(hx_n)\right| < \frac{N}{10}$. Show that every subinterval $I = [\alpha, \beta) \subset U$ with $\beta - \alpha \geqslant \frac{4}{k+1}$ satisfies

$$\#\{n < N : x_n \in I\} \geqslant \frac{N(\beta - \alpha)}{2}.$$

6. Let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. Formulate and prove a generalization of the Weyl criterion for the equidistribution a sequence $(x_n)_{n\geqslant 0}$ in T. Prove that if $\vec{\alpha} = (\alpha_1, \ldots, \alpha_d)$, then the sequence $x_n = n\vec{\alpha}$ is equidistributed if and only $1, \alpha_1, \ldots, \alpha_d$ are linearly independent over $\mathbb{Q}$. Now let $F = \mathbb{Z}/k\mathbb{Z}$ be a finite cyclic group, let μ be the product of the uniform measure on F and the Lebesgue measure on $\mathbb{T}^d$, and let $(f, \vec{\alpha})$ be an element of $F \times \mathbb{T}^d$. Give necessary and sufficient conditions for the sequence $x_n = (nf, n\vec{\alpha})_{n\geqslant 0}$ to be equidistributed in $F \times \mathbb{T}^d$ w.r.t. μ.

7. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an invertible matrix with a, c nonzero. Show that the lattice $A\mathbb{Z}^2$ is admissible if and only if both $\frac{b}{a}$ and $\frac{d}{c}$ are BA.

8. For $d \geqslant 1$, $h \in \mathbb{Z}^d \smallsetminus \{0\}$ and $\delta > 0$, show that the indefinite integral

$$J(h) = \int_{U^d} \left[\|\langle h, \alpha\rangle\| (\log \|\langle h, \alpha\rangle\|)^{1+\delta}\right]^{-1} d\mathrm{Vol}(\alpha)$$

converges and is bounded above by a number which may depend on δ but does not depend on h (Notation: here $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{R}^d$, $\|x\|$ is the distance of x to the nearest integer).

9. For a lattice Λ, the *dual lattice* Λ^* is defined by $\Lambda^* = \{x \in \mathbb{R}^d : \forall y \in \Lambda, \langle x, y\rangle \in \mathbb{Z}\}$. Show that if $\Lambda = A(\mathbb{Z}^d)$ for a real invertible matrix A, then $\Lambda^* = {}^{t}A^{-1}(\mathbb{Z}^d)$, and that if Λ is admissible, so is Λ^*.

Furthermore, $\mathrm{Nm}(\Lambda^*)$ is bounded below by a number depending only on $\mathrm{Nm}(\Lambda)$.

10. Let k be a totally real number field of degree d, let $\mathcal{O}_k$ denote the ring of integers in k. A number $\alpha \in \mathcal{O}_k$ is called a *unit* if its minimal polynomial is of the form $P(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_1 x \pm 1$, where $a_i \in \mathbb{Z}$. Prove that α is a unit if and only if both α and α^{-1} belong to $\mathcal{O}_k$, and thus that the units form a group under multiplication. Deduce that

$$\{a \in A : a\Lambda = \Lambda\}$$

is a cocompact subgroup of $A = \{\mathrm{diag}(e^{t_1}, \ldots, e^{t_d}) : \sum t_i = 0\}$, where Λ is the geometric embedding of $\mathcal{O}_k$ which was discussed in class. *Hint: use Dirichlet's theorem on units.*

11. Let $k, \mathcal{O}_k, \Lambda$ be as in Ex. 10. Show that there is $c > 0$, depending only on $\mathrm{Nm}(\Lambda)$, such that for every $\rho > 0$,

$$\sum_{h \in \Gamma^* \smallsetminus \{0\}, \|h\| \leqslant \rho} \frac{1}{|\mathrm{Nm}(h)|} \leqslant c(\log(3+\rho))^d.$$

12. Prove Roth's theorem: for each $d \geqslant 2$ there is $c > 0$ such that for any $x_0, \ldots, x_{N-1} \in U^d$, $D((x_n)_{n=0}^{N-1}; \mathcal{R}_d) \geqslant c(\log N)^{\frac{d-1}{2}}$.

13. Let $d \geqslant 2$ and let $\mathcal{A}_d$ denote the collection of convex subsets of U^d. Prove that there is $c > 0$ (depending on d) such that for any $(x_n)_{n=0}^{N-1}$ in U^d, $D((x_n)_{n=0}^{N-1}; \mathcal{A}_d) \geqslant cN^{1-\frac{2}{d+1}}$.

14. For the following choices of X and $\mathcal{S}$, prove the stated bounds for the dual shatter function $\pi^*_{\mathcal{S}}(m)$.

(a) $X = U^2$, $\mathcal{S}$ is the collection of intersections of X with closed half-planes; i.e.

$$\mathcal{S} = \left\{U \cap \{x : f^*(x) \geqslant c\} : c \in \mathbb{R},\ f^* \in (R^2)^* \smallsetminus \{0\}\right\}.$$

Here V^* is the collection of linear functionals on a vector space V. Show $\pi^*_{\mathcal{S}}(m) = O(m^2)$.

(b) $X = U^d$, $\mathcal{S}$ is the collection of intersections of X with closed half-spaces; i.e.

$$\mathcal{S} = \left\{U \cap \{x : f^*(x) \geqslant c\} : c \in \mathbb{R},\ f^* \in (R^d)^* \smallsetminus \{0\}\right\}.$$

Show $\pi^*_{\mathcal{S}}(m) = O(m^d)$.

(c) $X = U^2$, $\mathcal{S} = \left\{U^2 \cap \overline{B}(x,r) : x \in \mathbb{R}^2, r > 0\right\}$. Here $\overline{B}(x,r)$ is the closed ball with center x and radius r. Show $\pi^*_{\mathcal{S}}(m) = O(m^2)$.

(d) $X = U^2 \cong \mathbb{R}^2/\mathbb{Z}^2$, $\mathcal{S} = \left\{U^2 \cap \pi(\overline{B}(x,r)) : x \in \mathbb{R}^2, r \in (0,1/2)\right\}$. Here $\pi : \mathbb{R}^2 \to U^2$ is the natural projection, so that $\mathcal{S}$ is the collection of images of closed balls in the plane, for which the projection map is injective. Show $\pi^*_{\mathcal{S}}(m) = O(m^2)$.

15. Compute the VC-dimension of the following set systems $(X, \mathcal{S})$:

- $X = \mathbb{R}^2$, $\mathcal{S}$ is the collection of closed half-spaces in $\mathbb{R}^2$.
- $X = \mathbb{R}$, $\mathcal{S} = \{\{x : \sin(ax) \geqslant 0\} : a \in \mathbb{R}\}$.
- $X = U^d$, $\mathcal{S} = \mathcal{R}_d$.
- $X = \mathbb{R}^d$, $\mathcal{S}$ is the collection of closed boxes, that is set of the form $O([a_1, b_1] \times \cdots \times [a_d, b_d])$ where $a_i < b_i$ for $i = 1, \ldots, d$ and O is an orthogonal transformation of $\mathbb{R}^d$.

16. Show that for any $d \in \mathbb{N}$ there is a closed convex $C \subset \mathbb{R}^2$ such that the VC-dimension of the set system

$$\left(\mathbb{R}^2, \{\text{isometric copies of } C\}\right)$$

is at least d.