Exercise sheet – Introduction to Ergodic Theory. Tel Aviv University, Fall 2023

Definitions and notations.

- For $x \in \mathbb{R}$, we write $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x\}$.
- Given a set X, a collection $\mathcal{B} \subset 2^X$ is called an *algebra of sets* if it contains the empty set, and is closed under finite unions, intersections and complements. A function $\mu : \mathcal{B}$ to $[0, \infty]$ is called a *finitely additive probability* if $\mu(X) = 1$ and μ is additive on finite disjoint unions.

1.

(i) Let X be a set, $\mathcal{B} \subset 2^X$ an algebra of sets, μ a finitely additive probability on \mathcal{B} . Let $(A_n)_{n=0}^{\infty}$ be a sequence of sets in \mathcal{B} such that for any $n \ge m \ge 0$,

$$\mu(A_n \cap A_m) = \mu(A_0 \cap A_{n-m}),\tag{1}$$

and suppose $a = \mu(A_0) > 0$. Prove that there is an integer

$$k \leqslant \left\lfloor \frac{1}{a} \right\rfloor \tag{2}$$

such that $\mu(A_0 \cap A_k) > 0$. Prove that (2) is sharp, that is, the conclusion need not hold for smaller k.

(ii) Prove that for any $j \ge 1$ and any a > 0, there is $k = k(j, a) \ge 1$, such that for any $X, \mathcal{B}, \mu, (A_n)$ with $\mu(A_0) = a$ and satisfying

$$0 \leqslant i_1 \leqslant \cdots \leqslant i_j \leqslant k$$

 $\implies \mu(A_{i_1} \cap \dots \cap A_{i_j}) = \mu(A_0 \cap A_{i_2-i_1} \cap \dots \cap A_{i_j-i_1}),$ there are $1 \leq i_1 < i_2 < \dots < i_j \leq k$ with $\mu(A_0 \cap A_{i_1} \cap \dots \cap A_{i_j}) > 0$. Find the sharp value of k(j, a).

- (iii) Prove or disprove: for any $j \ge 1$ and any a > 0, there is $k = k(j, a) \ge 1$, such that for any $X, \mathcal{B}, \mu, (A_n)$ with $\mu(A_0) = a$ and satisfying (1), there are $i_1 < \cdots < i_j \le k$ with $\mu(A_0 \cap A_{i_1} \cap \cdots \cap A_{i_j}) > 0$.
- (iv) Show that for any $a, \lambda \in (0, 1)$ there is $c = c(\lambda, a) > 0$ such that for any $X, \mathcal{B}, \mu, (A_n)$ as in (i), with $a = \mu(A_0) > 0$, there is $k \in \mathbb{N} \cap (0, c)$ such that $\mu(A_0 \cap A_k) \ge \lambda a^2$. Show that this cannot be improved to $\lambda = 1$.

2. Let X be a compact metric space, let $T : X \to X$ a homeomorphism, and let $x_0 \in X$. The α -limit set (respectively, the ω -limit set) of x_0 is defined to be the set of all accumulation points of the sequence $\{T^n x_0 : n = -1, -2, \ldots\}$ (resp., of the sequence $\{T^n x_0 : n = 1, 2, \ldots\}$).

The wandering set is the set of all $x \in X$ for which there is a neighborhood U such that the sets $\{T^{-n}(U) : n \in \mathbb{N}\}$ are disjoint, and the non-wandering set is the complement of the wandering set.

- (i) Find an example of (T, X) and x_0 as above for which the α limit set and the ω limit set are distinct.
- (ii) Find an example as in (i) for which the wandering set is empty.
- (iii) Find an example of (T, X) and x_0 for which the wandering set is empty and for which the α -limit set is properly contained in the ω -limit set.
- (iv) Show that the non-wandering set of (T, X) is nonempty, *T*-invariant and compact. Show the same properties for the α -limit set and ω -limit set of any x_0 .
- (v) Let A_0 be the non-wandering set of (T, X), and inductively define A_{n+1} to be the non-wandering set of the restriction of Tto A_n . Give an example showing that $A_1 \neq A_0$. Show that any T-invariant Borel probability measure on K gives measure 1 to A_n for any n.

3. Let $\mathbf{a} \in \{0, 1\}^{\mathbb{N}}$ be the *Thue-Morse sequence*, defined as the limit of the following sequence of finite sequences:

0, 01, 0110, 01101001, 0110100110010110, ...

(at each stage, every appearance of 0 is replaced with 01 and every appearance of 1 is replaced with 10).

- Prove that using the above rule, every finite word is a prefix of every subsequent word, and thus the infinite sequence **a** is well-defined.
- Let $\mathbf{b} \in \{0,1\}^{\mathbb{Z}}$ denote the bi-infinite word whose entries on \mathbb{N} coincide with the entries of \mathbf{a} , and whose entries on $\{\ldots, -2, -1, 0\}$ are all 0. Let $T : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ denote the shift and let Ω denote the set of accumulation points of the sequence $\{T^n\mathbf{b}: n = 1, 2, \ldots\}$. Prove that Ω is a minimal subset of $(T, \{0,1\}^{\mathbb{Z}})$.

4. Let (G, +) be an abelian lcsc group and X a compact metric G-space. We say that a subset Y of G is syndetic if there is a compact subset $K \subset G$ such that G = Y + K. We say that the system (G, X) is pointwise minimal if for every $x \in X, \overline{Gx}$ is a minimal subset of X. Prove that (G, X) is pointwise minimal if and only if for any $x_0 \in X$ and any neighborhood V of x_0 , the set $\{g \in G : gx_0 \in V\}$ is syndetic.

5. Let G be an less group and let X be an less space such that G acts on X continuously.

- (1) Suppose there is $x_0 \in X$ such that the orbit Gx_0 is dense. Prove that the collection of points in X whose orbit is dense contains a dense G_{δ} subset of X.
- (2) Let μ be an ergodic quasi-invariant Borel measure, where *quasi-invariant* means that

A Borel, $\mu(A) = 0 \implies \forall g \in G, \mu(g(A)) = 0$,

and ergodic means

A Borel and G-invariant $\implies \mu(A) = 0$ or $\mu(G \setminus A) = 0$.

- Suppose $\mu(U) > 0$ for every non-empty open U. Prove that the orbit of a.e. $x_0 \in X$ is dense.
- Suppose Y is lcsc and $f : X \to Y$ is a G-invariant Borel map. Prove that f is constant a.e.
- **6.** Let (X, \mathcal{B}, μ, T) be a p.p.s. on a Borel probability space.
 - Prove that for any $B \in \mathcal{B}$ and any $\varepsilon > 0$, the set

$$\left\{n \in \mathbb{N} : \mu(T^{-n}(B) \cap B) \ge \mu(B)^2 - \varepsilon\right\}$$

has bounded gaps.

• Show that the p.p.s. is ergodic if and only if for any $A, B \in \mathcal{B}$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(T^{-n}(A) \cap B) = \mu(A)\mu(B).$$

• Give an example of an ergodic p.p.s. for which there are A, B in \mathcal{B} such that the sequence $\left(\frac{1}{N}\sum_{n=0}^{N-1} |\mu(T^{-n}(A) \cap B) - \mu(A)\mu(B)|\right)_{N \in \mathbb{N}}$ does not converge to 0 as $N \to \infty$.

7. Let $H \subset \mathbb{R}^n$ be a closed subgroup (with respect to vector addition), and let $H^{(0)}$ be the connected component of $\{0\}$ in H. Prove that $H^{(0)}$ is a vector subspace, and that there are linearly independent $v_1, \ldots, v_k \in H$, with $k \leq n - \dim H^{(0)}$, such that the group Λ generated by v_1, \ldots, v_k is discrete and $H = H^{(0)} \times \Lambda$. Let $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$, let m be the Haar measure on \mathbb{T}^n , let $v = (v_1, \ldots, v_n)$, and define

$$T: \mathbb{T}^n \to \mathbb{T}^n, \quad T(x) = x + v.$$

Prove that T preserves m, and that the following are equivalent.

- The p.p.s. (\mathbb{T}^n, m, T) is ergodic.
- The action of T on \mathbb{T}^n is minimal.
- *m* is the unique *T*-invariant Borel probability measure on \mathbb{T}^n .
- The numbers $1, v_1, \ldots, v_n$ are linearly independent over \mathbb{Q} .

In case the above conditions do not hold, show that the closure of $\{T^k x : k = 1, 2, ...\}$ is a submanifold of \mathbb{T}^n , and give a formula for its dimension.

8. Suppose $G = \mathbb{R}^2$ and $G_0 = \{(x, y) : x > 0, |y| \leq 0.1 x\}$. Let X be a compact metric space on which G acts. Say that $X_0 \subset X$ is G_0 -invariant if $g_0(X_0) \subset X_0$ for any $g_0 \in G_0$.

- (i) Show that any nonempty closed G_0 -invariant subset contains a nonempty closed G-invariant subset.
- (ii) Show that if G acts minimally on X then for any $x_0 \in X$, $G_0 x_0$ is dense in X.
- (iii) Show that (ii) may fail if G_0 is replaced with $\{(x, 0) : x > 0\}$.
- (iv) Give an example of X and X_0 which is closed, G_0 -invariant, and not G-invariant.

9. Let X be a noncompact lcsc space on which \mathbb{R} acts continuously. Denote the action by $(t, x) \mapsto \varphi^t(x)$. Suppose any orbit $\{\varphi^t(x) : t \in \mathbb{R}\}$ is dense. Prove that there is $x_0 \in X$ such that one of the two semi-orbits $\{\varphi^t(x_0) : t \leq 0\}, \{\varphi^t(x_0) : t \geq 0\}$ is not dense.

10. Let $\Omega = \{0,1\}^{\mathbb{Z}}$ and let $T : \Omega \to \Omega$ be the shift map. If $\omega \in \Omega$ such that $T^p \omega = \omega$ for some $p \in \mathbb{N}$, then we define the uniform measure on ω by $\mu = \frac{1}{p} \sum_{i=0}^{p-1} \delta_{T^i \omega}$. Show that the uniform measure on such a periodic orbit is *T*-invariant and ergodic, and show that uniform measures on periodic orbits are dense in the simplex $\operatorname{Prob}(\Omega)^T$ of invariant measures.

11. Let $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$, and let **Erg** denote the measures $\mu \in \operatorname{Prob}(\mathbb{T}^n)$ for which there is a subspace $V \subset \mathbb{R}^n$ such that μ is V-invariant and ergodic. Show that **Erg** is closed in $\operatorname{Prob}(\mathbb{T}^n)$.

12. Let X be a compact metric space, let $T : X \to X$ be a homeomorphism, and let $\mu \in \operatorname{Prob}(X)^T$. Show that the following are equivalent.

- $\{\mu\} = \operatorname{Prob}(X)^T$, i.e., the system is uniquely ergodic.
- For any $f \in C(X)$, the averages $\frac{1}{N} \sum_{n=0}^{N-1} f(T^n(x))$ converge to $\int_X f \, d\mu$ uniformly in x.
- T is ergodic with respect to μ , and for every $x \in X$ there is $C \ge 0$ such that for any non-negative $f \in C(X)$,

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n(x)) \leqslant C \int_X f \, d\mu.$$

13. Let $(a_n) = (1, 2, 4, 8, 1, 3, 6, 1, ...)$ be the sequence of first digits of the powers of 2. For each $k \in \{1, ..., 9\}$, show that

$$\lim_{N \to \infty} \frac{1}{N} \# \left\{ 1 \le n \le N : a_n = k \right\} = \log_{10} \left(\frac{k+1}{k} \right)$$

14. Let X be a compact metric space, $T : X \to X$ a homeomorphism, and \mathcal{E} the collection of T-invariant ergodic regular Borel measures on X. Let $f \in C(X)$ and define

$$m_f = \inf\left\{\int_X fd\mu : \mu \in \mathcal{E}\right\}, \quad M_f = \sup\left\{\int_X fd\mu : \mu \in \mathcal{E}\right\}.$$

Prove that for any $x \in X$,

$$m_f \leq \liminf_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x), \quad \limsup_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \leq M_f.$$

15. Suppose X is a compact metric space, G is an lcsc group acting continuously on X, $\mu \in \operatorname{Prob}(X)^G$ and ν_1, ν_2 are two probability measures on $\operatorname{Prob}(X)$ satisfying the following for i = 1, 2:

- $\nu_i \left(\left\{ \theta \in \operatorname{Prob}(X)^G : \theta \text{ is ergodic for the action of } G \right\} \right) = 1.$
- For all $f \in C(X)$, $\int_X f \, d\mu = \int_{\operatorname{Prob}(X)} \left[\int_X f \, d\theta \right] \, d\nu_i(\theta)$.

Prove that $\nu_1 = \nu_2$.

This is the uniqueness statement in the version of the ergodic decomposition theorem proved in the lecture.

16. Prove that the following groups are not amenable: $SL_2(\mathbb{R})$, with its topology as a subset of 2×2 real matrices, and F_r , the free group on $r \ge 2$ generators, with the discrete topology.

17. Let $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ be the upper half-plane, let $\Delta = \{z \in \mathbb{H} : |\operatorname{Re}(z)| \leq 1, |z - 1/2| \geq 1/2, |z + 1/2| \geq 1/2\}$, and let

$$\Gamma = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle.$$

Show that:

- Γ is of finite index in $SL_2(\mathbb{Z})$;
- Γ is isomorphic to F_2 , the free group on two generators;
- the translates $\{\gamma(\Delta) : \gamma \in \Gamma\}$ cover \mathbb{H} , where the action of Γ on \mathbb{H} is by Möbius transformations;
- these translates have disjoint interiors;
- Γ and $SL_2(\mathbb{Z})$ are non-uniform lattices in $SL_2(\mathbb{R})$.

18. For $x, y, z \in \mathbb{R}$, let

$$u_{x,y,z} = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

and let $G = \{u_{x,y,z} : x, y, z \in \mathbb{R}\}, \quad \Gamma = \{u_{x,y,z} : x, y, z \in \mathbb{Z}\}.$ Show that:

- G is a group with respect to matrix multiplication, and Γ is a non-normal subgroup;
- G is unimodular;
- Γ is a cocompact lattice;
- Let $u_0 = u_{a,b,c} \in G$, let $X = G/\Gamma$, and let $T : X \to X$ be the map $T(g\Gamma) = (u_0g)\Gamma$. Show that the following are equivalent:
 - -T is ergodic with respect to the *G*-invariant measure m_X .
 - -T is uniquely ergodic.
 - -1, a, b are linearly independent over \mathbb{Q} .

19. Let $\|\cdot\|$ denote the Euclidean norm \mathbb{R}^2 and let m denote the Lebesgue measure on \mathbb{R}^2 . Also let $\|\cdot\|'$ denote the Euclidean norm on $M_2(\mathbb{R}) \cong \mathbb{R}^4$. Let Γ be a cocompact lattice in $SL_2(\mathbb{R})$ and let $\Gamma_T = \{\gamma \in \Gamma : \|\gamma\|' \leq T\}$. Show that there is a constant C > 0 such that for any continuous compactly supported function f on $\mathbb{R}^2 \setminus \{0\}$ and any $v \in \mathbb{R}^2 \setminus \{0\}$ we have

$$\frac{1}{T}\sum_{\gamma\in\Gamma_T}f(\gamma v)\stackrel{T\to\infty}{\longrightarrow}\frac{C}{\|v\|}\int_{\mathbb{R}^2}\frac{f(x)}{\|x\|}\,dm(x).$$

20. Let $X = G/\Gamma$, where $G = SL_2(\mathbb{R})$, $\Gamma = SL_2(\mathbb{Z})$. Let $\pi : G \to X$ be the projection, let let

$$u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad g_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G,$$

and let $x_0 = \pi(g_0)$. Prove that the following are equivalent:

- The orbit $\{u_s x_0 : s \in \mathbb{R}\}$ is periodic, i.e., there is $s_0 > 0$ such that $u_{s_0} x_0 = x_0$.
- Either d = 0 or $\frac{c}{d} \in \mathbb{Q}$.
- The trajectory $\{a_{-t}x_0 : t \ge 0\}$ is divergent, i.e., for any compact $K \subset X$ there is t_0 such that for all $t > t_0$, $a_{-t}x_0 \notin K$.
- The orbit of the row vector $(c, d) \in \mathbb{R}^2$ under right matrix multiplication by elements of Γ , is discrete.