Definitions and notations.

- For $x \in \mathbb{R}$, we write $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x\}$. 
- Given a set $X$, a collection $\mathcal{B} \subset 2^X$ is called an algebra of sets if it contains the empty set, and is closed under finite unions, intersections and complements. A function $\mu : \mathcal{B} \to [0, \infty]$ is called a finitely additive probability if $\mu(X) = 1$ and $\mu$ is additive on finite disjoint unions.

1. Let $X$ be a set, $\mathcal{B} \subset 2^X$ an algebra of sets, $\mu$ a finitely additive probability on $\mathcal{B}$. Let $(A_n)_{n=0}^\infty$ be a sequence of sets in $\mathcal{B}$ such that for any $n \geq m \geq 0$,

$$\mu(A_n \cap A_m) = \mu(A_0 \cap A_{n-m}),$$

and suppose $a = \mu(A_0) > 0$. Prove that there is an integer

$$k \leq \frac{1}{a}$$

such that $\mu(A_0 \cap A_k) > 0$. Prove that (2) is sharp, that is, the conclusion need not hold for smaller $k$.

(ii) Prove that for any $j \geq 1$ and any $a > 0$, there is $k = k(j, a) \geq 1$, such that for any $X, \mathcal{B}, \mu, (A_n)$ with $\mu(A_0) = a$ and satisfying

$$0 \leq i_1 \leq \cdots \leq i_j \leq k$$

$$\implies \mu(A_{i_1} \cap \cdots \cap A_{i_j}) = \mu(A_0 \cap A_{i_2-i_1} \cap \cdots \cap A_{i_j-i_1}),$$

there are $1 \leq i_1 < i_2 < \cdots < i_j \leq k$ with $\mu(A_0 \cap A_{i_1} \cap \cdots \cap A_{i_j}) > 0$. Find the sharp value of $k(j, a)$.

(iii) Prove or disprove: for any $j \geq 1$ and any $a > 0$, there is $k = k(j, a) \geq 1$, such that for any $X, \mathcal{B}, \mu, (A_n)$ with $\mu(A_0) = a$ and satisfying (1), there are $i_1 < \cdots < i_j \leq k$ with $\mu(A_0 \cap A_{i_1} \cap \cdots \cap A_{i_j}) > 0$.

(iv) Show that for any $a, \lambda \in (0, 1)$ there is $c = c(\lambda, a) > 0$ such that for any $X, \mathcal{B}, \mu, (A_n)$ as in (i), with $a = \mu(A_0) > 0$, there is $k \in \mathbb{N} \cap (0, c)$ such that $\mu(A_0 \cap A_k) \geq \lambda a^2$. Show that this cannot be improved to $\lambda = 1$.

2. Let $X$ be a compact metric space, let $T : X \to X$ a homeomorphism, and let $x_0 \in X$. The $\alpha$-limit set (respectively, the $\omega$-limit set) of $x_0$ is defined to be the set of all accumulation points of the sequence $\{T^n x_0 : n = -1, -2, \ldots\}$ (resp., of the sequence $\{T^n x_0 : n = 1, 2, \ldots\}$).
The \textit{wandering set} is the set of all \( x \in X \) for which there is a neighborhood \( U \) such that the sets \( \{ T^{-n}(U) : n \in \mathbb{N} \} \) are disjoint, and the \textit{non-wandering set} is the complement of the wandering set.

(i) Find an example of \( (T, X) \) and \( x_0 \) as above for which the \( \alpha \) limit set and the \( \omega \) limit set are distinct.

(ii) Find an example as in (i) for which the wandering set is empty.

(iii) Find an example of \( (T, X) \) and \( x_0 \) for which the wandering set is empty and for which the \( \alpha \)-limit set is properly contained in the \( \omega \)-limit set.

(iv) Show that the non-wandering set of \( (T, X) \) is nonempty, \( T \)-invariant and compact. Show the same properties for the \( \alpha \)-limit set and \( \omega \)-limit set of any \( x_0 \).

(v) Let \( A_0 \) be the non-wandering set of \( (T, X) \), and inductively define \( A_{n+1} \) to be the non-wandering set of the restriction of \( T \) to \( A_n \). Given an example showing that \( A_1 \neq A_0 \). Show that any \( T \)-invariant Borel probability measure on \( K \) gives measure 1 to \( A_n \) for any \( n \).

3. Let \( a \in \{0,1\}^\mathbb{N} \) be the \textit{Thue-Morse sequence}, defined as the limit of the following sequence of finite sequences:

\[
0, 01, 0110, 01101001, 0110100110010110, \ldots
\]

(at each stage, every appearance of 0 is replaced with 01 and every appearance of 1 is replaced with 10).

- Prove that using the above rule, every finite word is a prefix of every subsequent word, and thus the infinite sequence \( a \) is well-defined.
- Let \( b \in \{0,1\}^\mathbb{Z} \) denote the bi-infinite word whose entries on \( \mathbb{N} \) coincide with the entries of \( a \), and whose entries on \( \{\ldots,-2,-1,0\} \) are all 0. Let \( T : \{0,1\}^\mathbb{Z} \to \{0,1\}^\mathbb{Z} \) denote the shift and let \( \Omega \) denote the set of accumulation points of the sequence \( \{T^n b : n = 1, 2, \ldots\} \). Prove that \( \Omega \) is a minimal subset of \( (T, \{0,1\}^\mathbb{Z}) \).

4. Let \( (G, +) \) be an abelian lcsc group and \( X \) a compact metric \( G \)-space. We say that a subset \( Y \) of \( G \) is \textit{syndetic} if there is a compact subset \( K \subset G \) such that \( G = Y + K \). We say that the system \( (G, X) \) is \textit{pointwise minimal} if for every \( x \in X \), \( \overline{Gx} \) is a minimal subset of \( X \). Prove that \( (G, X) \) is pointwise minimal if and only if for any \( x_0 \in X \) and any neighborhood \( V \) of \( x_0 \), the set \( \{ g \in G : gx_0 \in V \} \) is syndetic.

5. Let \( G \) be an lcsc group and let \( X \) be an lcsc space such that \( G \) acts on \( X \) continuously.
(1) Suppose there is \( x_0 \in X \) such that the orbit \( Gx_0 \) is dense. Prove that the collection of points in \( X \) whose orbit is dense contains a dense \( G_\delta \) subset of \( X \).

(2) Let \( \mu \) be an ergodic quasi-invariant Borel measure, where quasi-invariant means that

\[
\forall A \in \mathcal{B}, \mu(p_A q) = 0 \iff \mu(p_A q) = 0.
\]

and ergodic means

\[
\forall A \in \mathcal{B}, \mu(A) = 0 \iff \mu(G \Delta A) = 0.
\]

7. Let \( H \subset \mathbb{R}^n \) be a closed subgroup (with respect to vector addition), and let \( H^{(0)} \) be the connected component of \( \{0\} \) in \( H \). Prove that \( H^{(0)} \) is a vector subspace, and that there are linearly independent \( v_1, \ldots, v_k \in H \), with \( k \leq n - \dim H^{(0)} \), such that the group \( \Lambda \) generated by \( v_1, \ldots, v_k \) is discrete and \( H = H^{(0)} \times \Lambda \). Let \( \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n \), let \( m \) be the Haar measure on \( \mathbb{T}^n \), let \( v = (v_1, \ldots, v_n) \), and define

\[
T : \mathbb{T}^n \to \mathbb{T}^n, \quad T(x) = x + v.
\]

Prove that \( T \) preserves \( m \), and that the following are equivalent.

- The p.p.s. \( (\mathbb{T}^n, m, T) \) is ergodic.
- The action of \( T \) on \( \mathbb{T}^n \) is minimal.
- \( m \) is the unique \( T \)-invariant Borel probability measure on \( \mathbb{T}^n \).
- The numbers 1, \( v_1, \ldots, v_n \) are linearly independent over \( \mathbb{Q} \).
In case the above conditions do not hold, show that the closure of 
\( \{ T^k x : k = 1, 2, \ldots \} \) is a submanifold of \( \mathbb{T}^n \), and give a formula for its dimension.

8. Suppose \( G = \mathbb{R}^2 \) and \( G_0 = \{ (x, y) : x > 0, |y| \leq 0.1 x \} \). Let \( X \) be a compact metric space on which \( G \) acts. Say that \( X_0 \subset X \) is \( G_0 \)-invariant if \( g_0(X_0) \subset X_0 \) for any \( g_0 \in G_0 \).

(i) Show that any closed \( G_0 \)-invariant subset contains a closed \( G \)-invariant subset.

(ii) Show that if \( G \) acts minimally on \( X \) then for any \( x_0 \in X \), \( G_0 x_0 \) is dense in \( X \).

(iii) Show that (ii) may fail if \( G_0 \) is replaced with \( \{ (x, 0) : x > 0 \} \).

(iv) Give an example of \( X \) and \( X_0 \) which is closed, \( G_0 \)-invariant, and not \( G \)-invariant.

9. Let \( X \) be a noncompact lcsc space on which \( \mathbb{R} \) acts continuously. Denote the action by \( (t, x) \mapsto \varphi^t(x) \). Suppose any orbit \( \{ \varphi^t(x) : t \in \mathbb{R} \} \) is dense. Prove that there is \( x_0 \in X \) such that one of the two semi-orbits \( \{ \varphi^t(x_0) : t \leq 0 \} \), \( \{ \varphi^t(x_0) : t \geq 0 \} \) is not dense.

10. Let \( \Omega = \{0, 1 \}^\mathbb{Z} \) and let \( T : \Omega \to \Omega \) be the shift map. If \( \omega \in \Omega \) such that \( T^n \omega = \omega \) for some \( p \in \mathbb{N} \), then we define the uniform measure on \( \omega \) by \( \mu = \frac{1}{p} \sum_{i=0}^{p-1} \delta_{T^i \omega} \). Show that the uniform measure on such a periodic orbit is \( T \)-invariant and ergodic, and show that uniform measures on periodic orbits are dense in the simplex \( \text{Prob}(\Omega)^T \) of invariant measures.

11. Let \( \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n \), and let \( \text{Erg} \) denote the measures \( \mu \in \text{Prob}(\mathbb{T}^n) \) for which there is a subspace \( V \subset \mathbb{R}^n \) such that \( \mu \) is \( V \)-invariant and ergodic. Show that \( \text{Erg} \) is closed in \( \text{Prob}(\mathbb{T}^n) \).

12. Let \( X \) be a compact metric space, let \( T : X \to X \) be a homeomorphism, and let \( \mu \in \text{Prob}(X)^T \). Show that the following are equivalent.

- \( \{ \mu \} = \text{Prob}(X)^T \), i.e., the system is uniquely ergodic.
- For any \( f \in C(X) \), the averages \( \frac{1}{N} \sum_{n=0}^{N-1} f(T^n(x)) \) converge to \( \int_X f \, d\mu \) uniformly in \( x \).
- \( T \) is ergodic with respect to \( \mu \), and for every \( x \in X \) there is \( C \geq 0 \) such that for any non-negative \( f \in C(X) \),
  \[
  \limsup_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n(x)) \leq C \int_X f \, d\mu.
  \]
13. Let \( (a_n) = (1, 2, 4, 8, 1, 3, 6, 1, \ldots) \) be the sequence of first digits of the powers of 2. For each \( k \in \{1, \ldots, 9\} \), show that
\[
\lim_{N \to \infty} \frac{1}{N} \# \{1 \leq n \leq N : a_n = k\} = \log_{10} \left(\frac{k+1}{k}\right).
\]

14. Let \( X \) be a compact metric space, \( T : X \to X \) a homeomorphism, and \( \mathcal{E} \) the collection of \( T \)-invariant ergodic regular Borel measures on \( X \). Let \( f \in C(X) \) and define
\[
m_f = \inf \left\{ \int_X f d\mu : \mu \in \mathcal{E} \right\}, \quad M_f = \sup \left\{ \int_X f d\mu : \mu \in \mathcal{E} \right\}.
\]
Prove that for any \( x \in X \),
\[
m_f \leq \lim \inf_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x), \quad \lim \sup_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \leq M_f.
\]

15. Suppose \( X \) is a compact metric space, \( G \) is an lcsc group acting continuously on \( X \), \( \mu \in \text{Prob}(X)^G \) and \( \nu_1, \nu_2 \) are two probability measures on \( \text{Prob}(X) \) satisfying the following for \( i = 1, 2 \):
- \( \nu_i \left( \{ \theta \in \text{Prob}(X)^G : \theta \text{ is ergodic for the action of } G \} \right) = 1 \).
- For all \( f \in C(X) \), \( \int_X f d\mu = \int_{\text{Prob}(X)} \left[ \int_X f d\theta \right] d\nu_i(\theta) \).
Prove that \( \nu_1 = \nu_2 \).
This is the uniqueness statement in the version of the ergodic decomposition theorem proved in the lecture.

16. Prove that the following groups are not amenable: \( \text{SL}_2(\mathbb{R}) \), with its topology as a subset of \( 2 \times 2 \) real matrices, and \( F_r \), the free group on \( r \geq 2 \) generators, with the discrete topology.

17. Let \( \mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\} \) be the upper half-plane, let \( \Delta = \{z \in \mathbb{H} : |\text{Re}(z)| \leq 1, |z| \geq 1\} \), and let
\[
\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : a, d \text{ are odd and } b, c \text{ are even} \right\}.
\]
Show that:
- \( \Gamma \) is of finite index in \( \text{SL}_2(\mathbb{Z}) \);
- \( \Gamma \) is isomorphic to \( F_2 \), the free group on two generators;
- the translates \( \{\gamma(\Delta) : \gamma \in \Gamma\} \) covers \( \mathbb{H} \), where the action of \( \Gamma \) on \( \mathbb{H} \) is by Möbius transformations;
- these translates have disjoint interiors;
- \( \Gamma \) and \( \text{SL}_2(\mathbb{Z}) \) are non-uniform lattices in \( \text{SL}_2(\mathbb{R}) \).
18. For $x, y, z \in \mathbb{R}$, let
\[ u_{x,y,z} = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \]
and let $G = \{u_{x,y,z} : x, y, z \in \mathbb{R}\}$, $\Gamma = \{u_{x,y,z} : x, y, z \in \mathbb{Z}\}$. Show that:
- $G$ is a group with respect to matrix multiplication, and $\Gamma$ is a non-normal subgroup;
- $G$ is unimodular;
- $\Gamma$ is a cocompact lattice;
- Let $u_0 = u_{a,b,c} \in G$, let $X = G/\Gamma$, and let $T : X \to X$ be the map $T(g\Gamma) = (u_0g)\Gamma$. Show that the following are equivalent:
  - $T$ is ergodic with respect to the $G$-invariant measure $m_X$.
  - $T$ is uniquely ergodic.
  - $1, a, b$ are linearly independent over $\mathbb{Q}$.

19. Let $\| \cdot \|$ denote the Euclidean norm $\mathbb{R}^2$ and let $m$ denote the Lebesgue measure on $\mathbb{R}^2$. Also let $\| \cdot \|$ denote the Euclidean norm on $M_2(\mathbb{R}) \cong \mathbb{R}^4$. Let $\Gamma$ be a cocompact lattice in $\text{SL}_2(\mathbb{R})$ and let $\Gamma_T = \{\gamma \in \Gamma : \|\gamma\| \leq T\}$. Show that there is a constant $C > 0$ such that for any continuous compactly supported function $f$ on $\mathbb{R}^2 \setminus \{0\}$ and any $v \in \mathbb{R}^2 \setminus \{0\}$ we have
\[ \frac{1}{T} \sum_{\gamma \in \Gamma_T} f(\gamma v) \xrightarrow{T \to \infty} \frac{C}{\|v\|} \int_{\mathbb{R}^2} \frac{f(x)}{\|x\|} \, dm(x). \]

20. Let $X = G/\Gamma$, where $G = \text{SL}_2(\mathbb{R})$, $\Gamma = \text{SL}_2(\mathbb{Z})$. Let $\pi : G \to X$ be the projection, let
\[ u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad g_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, \]
and let $x_0 = \pi(g_0)$. Prove that the following are equivalent:
- The orbit $\{u_sx_0 : s \in \mathbb{R}\}$ is periodic, i.e., there is $s_0 > 0$ such that $u_{s_0}x_0 = x_0$.
- Either $d = 0$ or $\frac{c}{d} \in \mathbb{Q}$.
- The trajectory $\{a_{-t}x_0 : t \leq 0\}$ is divergent, i.e., for any compact $K \subset X$ there is $t_0$ such that for all $t > t_0$, $a_{-t}x_0 \not\in K$.
- The orbit of the row vector $(c, d) \in \mathbb{R}^2$ under right matrix multiplication by elements of $\Gamma$, is discrete.