**Exercise sheet – Introduction to Ergodic Theory.**
**Tel Aviv University, Fall 2023**

**Definitions and notations.**

- For $x \in \mathbb{R}$, we write $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leqslant x\}$.
- Given a set $X$, a collection $\mathcal{B} \subset 2^X$ is called an *algebra of sets* if it contains the empty set, and is closed under finite unions, intersections and complements. A function $\mu : \mathcal{B}$ to $[0, \infty]$ is called a *finitely additive probability* if $\mu(X) = 1$ and $\mu$ is additive on finite disjoint unions.

**1.**

(i) Let $X$ be a set, $\mathcal{B} \subset 2^X$ an algebra of sets, $\mu$ a *finitely additive probability* on $\mathcal{B}$. Let $(A_n)_{n=0}^{\infty}$ be a sequence of sets in $\mathcal{B}$ such that for any $n \geqslant m \geqslant 0$,

$$\mu(A_n \cap A_m) = \mu(A_0 \cap A_{n-m}), \tag{1}$$

and suppose $a = \mu(A_0) > 0$. Prove that there is an integer

$$k \leqslant \left\lfloor \frac{1}{a} \right\rfloor \tag{2}$$

such that $\mu(A_0 \cap A_k) > 0$. Prove that (2) is sharp, that is, the conclusion need not hold for smaller $k$.

(ii) Prove that for any $j \geqslant 1$ and any $a > 0$, there is $k = k(j, a) \geqslant 1$, such that for any $X, \mathcal{B}, \mu, (A_n)$ with $\mu(A_0) = a$ and satisfying

$$0 \leqslant i_1 \leqslant \cdots \leqslant i_j \leqslant k$$
$$\implies \quad \mu(A_{i_1} \cap \cdots \cap A_{i_j}) = \mu(A_0 \cap A_{i_2 - i_1} \cap \cdots \cap A_{i_j - i_1}),$$

there are $1 \leqslant i_1 < i_2 < \cdots < i_j \leqslant k$ with $\mu(A_0 \cap A_{i_1} \cap \cdots \cap A_{i_j}) > 0$. Find the sharp value of $k(j, a)$.

(iii) Prove or disprove: for any $j \geqslant 1$ and any $a > 0$, there is $k = k(j, a) \geqslant 1$, such that for any $X, \mathcal{B}, \mu, (A_n)$ with $\mu(A_0) = a$ and satisfying (1), there are $i_1 < \cdots < i_j \leqslant k$ with $\mu(A_0 \cap A_{i_1} \cap \cdots \cap A_{i_j}) > 0$.

(iv) Show that for any $a, \lambda \in (0, 1)$ there is $c = c(\lambda, a) > 0$ such that for any $X, \mathcal{B}, \mu, (A_n)$ as in (i), with $a = \mu(A_0) > 0$, there is $k \in \mathbb{N} \cap (0, c)$ such that $\mu(A_0 \cap A_k) \geqslant \lambda a^2$. Show that this cannot be improved to $\lambda = 1$.

**2.** Let $X$ be a compact metric space, let $T : X \to X$ a homeomorphism, and let $x_0 \in X$. The *$\alpha$-limit set* (respectively, the *$\omega$-limit set*) of $x_0$ is defined to be the set of all accumulation points of the sequence $\{T^n x_0 : n = -1, -2, \ldots\}$ (resp., of the sequence $\{T^n x_0 : n = 1, 2, \ldots\}$).

The *wandering set* is the set of all $x \in X$ for which there is a neighborhood $U$ such that the sets $\{T^{-n}(U) : n \in \mathbb{N}\}$ are disjoint, and the *non-wandering set* is the complement of the wandering set.

(i) Find an example of $(T, X)$ and $x_0$ as above for which the $\alpha$ limit set and the $\omega$ limit set are distinct.
(ii) Find an example as in (i) for which the wandering set is empty.
(iii) Find an example of $(T, X)$ and $x_0$ for which the wandering set is empty and for which the $\alpha$-limit set is properly contained in the $\omega$-limit set.
(iv) Show that the non-wandering set of $(T, X)$ is nonempty, $T$-invariant and compact. Show the same properties for the $\alpha$-limit set and $\omega$-limit set of any $x_0$.
(v) Let $A_0$ be the non-wandering set of $(T, X)$, and inductively define $A_{n+1}$ to be the non-wandering set of the restriction of $T$ to $A_n$. Give an example showing that $A_1 \neq A_0$. Show that any $T$-invariant Borel probability measure on $K$ gives measure 1 to $A_n$ for any $n$.

**3.** Let $\mathbf{a} \in \{0, 1\}^{\mathbb{N}}$ be the *Thue-Morse sequence*, defined as the limit of the following sequence of finite sequences:

$$0, 01, 0110, 01101001, 0110100110010110, \ldots$$

(at each stage, every appearance of 0 is replaced with 01 and every appearance of 1 is replaced with 10).

- Prove that using the above rule, every finite word is a prefix of every subsequent word, and thus the infinite sequence $\mathbf{a}$ is well-defined.
- Let $\mathbf{b} \in \{0, 1\}^{\mathbb{Z}}$ denote the bi-infinite word whose entries on $\mathbb{N}$ coincide with the entries of $\mathbf{a}$, and whose entries on $\{\ldots, -2, -1, 0\}$ are all 0. Let $T : \{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$ denote the shift and let $\Omega$ denote the set of accumulation points of the sequence $\{T^n\mathbf{b} : n = 1, 2, \ldots\}$. Prove that $\Omega$ is a minimal subset of $\big(T, \{0, 1\}^{\mathbb{Z}}\big)$.

**4.** Let $(G, +)$ be an abelian lcsc group and $X$ a compact metric $G$-space. We say that a subset $Y$ of $G$ is *syndetic* if there is a compact subset $K \subset G$ such that $G = Y + K$. We say that the system $(G, X)$ is *pointwise minimal* if for every $x \in X, \overline{Gx}$ is a minimal subset of $X$. Prove that $(G, X)$ is pointwise minimal if and only if for any $x_0 \in X$ and any neighborhood $V$ of $x_0$, the set $\{g \in G : gx_0 \in V\}$ is syndetic.

**5.** Let $G$ be an lcsc group and let $X$ be an lcsc space such that $G$ acts on $X$ continuously.

(1) Suppose there is $x_0 \in X$ such that the orbit $Gx_0$ is dense. Prove that the collection of points in $X$ whose orbit is dense contains a dense $G_\delta$ subset of $X$.

(2) Let $\mu$ be an ergodic quasi-invariant Borel measure, where *quasi-invariant* means that

$$A \text{ Borel}, \mu(A) = 0 \implies \forall g \in G, \mu(g(A)) = 0,$$

and *ergodic* means

$$A \text{ Borel and } G\text{-invariant} \implies \mu(A) = 0 \text{ or } \mu(G \smallsetminus A) = 0.$$

- Suppose $\mu(U) > 0$ for every non-empty open $U$. Prove that the orbit of a.e. $x_0 \in X$ is dense.
- Suppose $Y$ is lcsc and $f : X \to Y$ is a $G$-invariant Borel map. Prove that $f$ is constant a.e.

**6.** Let $(X, \mathcal{B}, \mu, T)$ be a p.p.s. on a Borel probability space.

- Prove that for any $B \in \mathcal{B}$ and any $\varepsilon > 0$, the set
$$\left\{n \in \mathbb{N} : \mu(T^{-n}(B) \cap B) \geqslant \mu(B)^2 - \varepsilon\right\}$$
has bounded gaps.
- Show that the p.p.s. is ergodic if and only if for any $A, B \in \mathcal{B}$,
$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(T^{-n}(A) \cap B) = \mu(A)\mu(B).$$
- Give an example of an ergodic p.p.s. for which there are $A, B$ in $\mathcal{B}$ such that the sequence $\left(\frac{1}{N}\sum_{n=0}^{N-1} |\mu\left(T^{-n}(A) \cap B\right) - \mu(A)\mu(B)|\right)_{N\in\mathbb{N}}$ does not converge to 0 as $N \to \infty$.

**7.** Let $H \subset \mathbb{R}^n$ be a closed subgroup (with respect to vector addition), and let $H^{(0)}$ be the connected component of $\{0\}$ in $H$. Prove that $H^{(0)}$ is a vector subspace, and that there are linearly independent $v_1, \ldots, v_k \in H$, with $k \leqslant n - \dim H^{(0)}$, such that the group $\Lambda$ generated by $v_1, \ldots, v_k$ is discrete and $H = H^{(0)} \times \Lambda$. Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, let $m$ be the Haar measure on $\mathbb{T}^n$, let $v = (v_1, \ldots, v_n)$, and define

$$T : \mathbb{T}^n \to \mathbb{T}^n, \quad T(x) = x + v.$$

Prove that $T$ preserves $m$, and that the following are equivalent.

- The p.p.s. $(\mathbb{T}^n, m, T)$ is ergodic.
- The action of $T$ on $\mathbb{T}^n$ is minimal.
- $m$ is the unique $T$-invariant Borel probability measure on $\mathbb{T}^n$.
- The numbers $1, v_1, \ldots, v_n$ are linearly independent over $\mathbb{Q}$.

In case the above conditions do not hold, show that the closure of $\{T^k x : k = 1, 2, \ldots\}$ is a submanifold of $\mathbb{T}^n$, and give a formula for its dimension.

**8.** Suppose $G = \mathbb{R}^2$ and $G_0 = \{(x, y) : x > 0, |y| \leqslant 0.1\,x\}$. Let $X$ be a compact metric space on which $G$ acts. Say that $X_0 \subset X$ is $G_0$-invariant if $g_0(X_0) \subset X_0$ for any $g_0 \in G_0$.

(i) Show that any nonempty closed $G_0$-invariant subset contains a nonempty closed $G$-invariant subset.
(ii) Show that if $G$ acts minimally on $X$ then for any $x_0 \in X$, $G_0 x_0$ is dense in $X$.
(iii) Show that (ii) may fail if $G_0$ is replaced with $\{(x, 0) : x > 0\}$.
(iv) Give an example of $X$ and $X_0$ which is closed, $G_0$-invariant, and not $G$-invariant.

**9.** Let $X$ be a noncompact lcsc space on which $\mathbb{R}$ acts continuously. Denote the action by $(t, x) \mapsto \varphi^t(x)$. Suppose any orbit $\{\varphi^t(x) : t \in \mathbb{R}\}$ is dense. Prove that there is $x_0 \in X$ such that one of the two semi-orbits $\{\varphi^t(x_0) : t \leqslant 0\}$, $\{\varphi^t(x_0) : t \geqslant 0\}$ is not dense.

**10.** Let $\Omega = \{0, 1\}^{\mathbb{Z}}$ and let $T : \Omega \to \Omega$ be the shift map. If $\omega \in \Omega$ such that $T^p \omega = \omega$ for some $p \in \mathbb{N}$, then we define the uniform measure on $\omega$ by $\mu = \frac{1}{p} \sum_{i=0}^{p-1} \delta_{T^i \omega}$. Show that the uniform measure on such a periodic orbit is $T$-invariant and ergodic, and show that uniform measures on periodic orbits are dense in the simplex $\mathrm{Prob}(\Omega)^T$ of invariant measures.

**11.** Let $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$, and let **Erg** denote the measures $\mu \in \mathrm{Prob}(\mathbb{T}^n)$ for which there is a subspace $V \subset \mathbb{R}^n$ such that $\mu$ is $V$-invariant and ergodic. Show that **Erg** is closed in $\mathrm{Prob}(\mathbb{T}^n)$.

**12.** Let $X$ be a compact metric space, let $T : X \to X$ be a homeomorphism, and let $\mu \in \mathrm{Prob}(X)^T$. Show that the following are equivalent.

- $\{\mu\} = \mathrm{Prob}(X)^T$, i.e., the system is uniquely ergodic.
- For any $f \in C(X)$, the averages $\frac{1}{N} \sum_{n=0}^{N-1} f(T^n(x))$ converge to $\int_X f\, d\mu$ uniformly in $x$.
- $T$ is ergodic with respect to $\mu$, and for every $x \in X$ there is $C \geqslant 0$ such that for any non-negative $f \in C(X)$,

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n(x)) \leqslant C \int_X f\, d\mu.$$

**13.** Let $(a_n) = (1, 2, 4, 8, 1, 3, 6, 1, \dots)$ be the sequence of first digits of the powers of 2. For each $k \in \{1, \dots, 9\}$, show that

$$\lim_{N\to\infty} \frac{1}{N} \# \{1 \leqslant n \leqslant N : a_n = k\} = \log_{10}\left(\frac{k+1}{k}\right).$$

**14.** Let $X$ be a compact metric space, $T : X \to X$ a homeomorphism, and $\mathcal{E}$ the collection of $T$-invariant ergodic regular Borel measures on $X$. Let $f \in C(X)$ and define

$$m_f = \inf\left\{\int_X f d\mu : \mu \in \mathcal{E}\right\}, \quad M_f = \sup\left\{\int_X f d\mu : \mu \in \mathcal{E}\right\}.$$

Prove that for any $x \in X$,

$$m_f \leqslant \liminf_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(T^n x), \quad \limsup_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(T^n x) \leqslant M_f.$$

**15.** Suppose $X$ is a compact metric space, $G$ is an lcsc group acting continuously on $X$, $\mu \in \mathrm{Prob}(X)^G$ and $\nu_1, \nu_2$ are two probability measures on $\mathrm{Prob}(X)$ satisfying the following for $i = 1, 2$:

- $\nu_i\left(\{\theta \in \mathrm{Prob}(X)^G : \theta \text{ is ergodic for the action of } G\}\right) = 1.$
- For all $f \in C(X)$, $\quad \int_X f\, d\mu = \int_{\mathrm{Prob}(X)} \left[\int_X f\, d\theta\right] d\nu_i(\theta).$

Prove that $\nu_1 = \nu_2$.

This is the uniqueness statement in the version of the ergodic decomposition theorem proved in the lecture.

**16.** Prove that the following groups are not amenable: $\mathrm{SL}_2(\mathbb{R})$, with its topology as a subset of $2 \times 2$ real matrices, and $F_r$, the free group on $r \geqslant 2$ generators, with the discrete topology.

**17.** Let $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the upper half-plane, let $\Delta = \{z \in \mathbb{H} : |\mathrm{Re}(z)| \leqslant 1,\ |z| \geqslant 1\}$, and let

$$\Gamma = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle.$$

Show that:

- $\Gamma$ is of finite index in $\mathrm{SL}_2(\mathbb{Z})$;
- $\Gamma$ is isomorphic to $F_2$, the free group on two generators;
- the translates $\{\gamma(\Delta) : \gamma \in \Gamma\}$ covers $\mathbb{H}$, where the action of $\Gamma$ on $\mathbb{H}$ is by Möbius transformations;
- these translates have disjoint interiors;
- $\Gamma$ and $\mathrm{SL}_2(\mathbb{Z})$ are non-uniform lattices in $\mathrm{SL}_2(\mathbb{R})$.

**18.** For $x, y, z \in \mathbb{R}$, let

$$u_{x,y,z} = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

and let $G = \{u_{x,y,z} : x, y, z \in \mathbb{R}\}, \quad \Gamma = \{u_{x,y,z} : x, y, z \in \mathbb{Z}\}$. Show that:

- $G$ is a group with respect to matrix multiplication, and $\Gamma$ is a non-normal subgroup;
- $G$ is unimodular;
- $\Gamma$ is a cocompact lattice;
- Let $u_0 = u_{a,b,c} \in G$, let $X = G/\Gamma$, and let $T : X \to X$ be the map $T(g\Gamma) = (u_0 g)\Gamma$. Show that the following are equivalent:
  - $T$ is ergodic with respect to the $G$-invariant measure $m_X$.
  - $T$ is uniquely ergodic.
  - $1, a, b$ are linearly independent over $\mathbb{Q}$.

**19.** Let $\|\cdot\|$ denote the Euclidean norm $\mathbb{R}^2$ and let $m$ denote the Lebesgue measure on $\mathbb{R}^2$. Also let $\|\cdot\|'$ denote the Euclidean norm on $M_2(\mathbb{R}) \cong \mathbb{R}^4$. Let $\Gamma$ be a cocompact lattice in $\mathrm{SL}_2(\mathbb{R})$ and let $\Gamma_T = \{\gamma \in \Gamma : \|\gamma\|' \leqslant T\}$. Show that there is a constant $C > 0$ such that for any continuous compactly supported function $f$ on $\mathbb{R}^2 \smallsetminus \{0\}$ and any $v \in \mathbb{R}^2 \smallsetminus \{0\}$ we have

$$\frac{1}{T} \sum_{\gamma \in \Gamma_T} f(\gamma v) \xrightarrow{T \to \infty} \frac{C}{\|v\|} \int_{\mathbb{R}^2} \frac{f(x)}{\|x\|} \, dm(x).$$

**20.** Let $X = G/\Gamma$, where $G = \mathrm{SL}_2(\mathbb{R}), \ \Gamma = \mathrm{SL}_2(\mathbb{Z})$. Let $\pi : G \to X$ be the projection, let let

$$u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad g_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G,$$

and let $x_0 = \pi(g_0)$. Prove that the following are equivalent:

- The orbit $\{u_s x_0 : s \in \mathbb{R}\}$ is periodic, i.e., there is $s_0 > 0$ such that $u_{s_0} x_0 = x_0$.
- Either $d = 0$ or $\frac{c}{d} \in \mathbb{Q}$.
- The trajectory $\{a_{-t} x_0 : t \geqslant 0\}$ is divergent, i.e., for any compact $K \subset X$ there is $t_0$ such that for all $t > t_0, \ a_{-t} x_0 \notin K$.
- The orbit of the row vector $(c, d) \in \mathbb{R}^2$ under right matrix multiplication by elements of $\Gamma$, is discrete.