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On Poincaré's Theorem for Fundamental Polygons

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Poincaré's classical theorem on fundamental polygons [2] gives sufficient conditions for a polygon to be a fundamental domain for a Fuchsian group. There are several published proofs of this theorem, but there is some question as to their validity; Siegel [4, p. 115] has commented on this and given an apparently valid proof under fairly restrictive conditions. None of the published proofs are as general as they might be, and they all have a convexity condition that is never really used.

This note is an attempt to clarify the situation. The problem and the solution presented below arose during the course of several informal conversations. Present at one or more of these conversations were L. V. Ahlfors, L. Bers, W. Magnus, J. E. McMillan, and B. Maskit.

Poincaré [3] also published a generalization of this theorem. The generalization is to polyhedra in 3-dimensional hyperbolic space, where the discontinuous group is Kleinian rather than Fuchsian. The recent work of Albert Marden [1] shows the importance of these polyhedra for the study of Kleinian groups, and so I have appended a statement and proof of Poincaré's polyhedron theorem.

1. In what follows, unless specifically stated otherwise, all sets are subsets of the unit disc U. The topology is always the relative topology, so that for a set S, $\bar{S}$ is the relative closure of S in U and ∂S is the relative boundary of S in U. Likewise the geometry is always the classical non-Euclidean geometry, where the geodesics, which we call *lines*, are arcs of circles orthogonal to the boundary of U. We recall that in this geometry, U is a complete unbounded metric space; we denote the distance between z and z' by $\rho(z, z')$.

Let G be a group of isometries of U (the elements of G are not

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necessarily orientation preserving). Two points z and z' are *equivalent* under G if there is a $g \in G$ so that $g(z) = z'$. G is called *discontinuous* if there is a nonempty open set V so that no two distinct points of V are equivalent under G.

It is well known that G is discontinuous if and only if G is a discrete subgroup of the group of all isometries of U. In fact, if G is discontinuous, then except for a discrete set of points, every point has a neighborhood V satisfying the above condition.

A set D is called a *fundamental domain* for the discontinuous group G if

(1) D is a domain, i.e., a non-empty connected open set,

(2) No two distinct points of D are equivalent under G, and

(3) Every point is equivalent under G to some point of $\bar{D}$.

If the boundary of the fundamental domain D is sufficiently nice, then we can fold together equivalent pieces of the boundary to get a representation of U/G, the space of equivalence classes modulo G. The nicest possibility is for ∂D to be piecewise linear, in which case we would call D a polygon. We now give the formal definition.

Let C be a straight line. A closed connected subset C' of C is *degenerate* if $C' = \varnothing$ or C' contains only one point.

A domain D is called a *polygon* if ∂D is a countable union of *sides* s_i, where each s_i is a nondegenerate connected closed subset of some line, and only finitely many sides meet any compact set. We require further that if $s_i \cap s_j \neq \varnothing$, $i \neq j$, then $s_i \cap s_j$ is a single point z, called a *vertex*, and z is an endpoint of both s_i and s_j. Finally, if any side s_i has a finite endpoint z, then there is exactly one other side s_j, where z is also an endpoint of s_j.

If D is a fundamental domain for G, and D is also a polygon, then D is called a *fundamental polygon* for G.

It is well known that every discontinuous group has a fundamental polygon. The simplest construction is to take some point 0 and let D be the set of all points which are closer to 0 than to any point of the form $g(0)$, $g \in G$. It is not hard to prove that D so constructed is a fundamental polygon for G. One also easily sees that the sides of D are pairwise identified by elements of G; these identifying elements in fact generate G.

2. Our problem here goes the other way. We have to start with a polygon D, satisfying certain conditions, and prove that it is a fundamental polygon for some group G. It is fairly obvious that the sides of D will have to be pairwise identified by elements of G, and these elements will generate G. We formalize this as follows.

An *identification* on the polygon D is a map which assigns, to each side s, a side s' and an isometry $A(s, s')$ so that

(a) $A(s, s')$ maps s onto s',

(b) $(s')' = s$ and $A(s', s) = (A(s, s'))^{-1}$,

(c) if $s = s'$, then $A(s, s')$ is the identity on s, and

(d) for each side s, there is a neighborhood V of s so that, setting $A(s, s') = A$, $A(V \cap D) \cap D = \varnothing$.

The isometries $A(s, s')$ which identify the sides of the polygon D are called *generators*, and the group generated by these generators is denoted by G.

If for some side s we have $s = s'$, then the above conditions imply that the corresponding generator $A = A(s, s')$ is of order two; in fact, A is then a reflection in the side s. These relations of the form $A^2 = 1$ are called the *reflection relations*.

We have to impose conditions on D so that it is a fundamental polygon for G. There are basically two conditions. The vertex condition makes note of the fact that we are dealing with conformal maps; i.e., the projection map from U to U/G should be conformal or integrally branched. Hence when we fold together $\bar{D}$ to get U/G we have to know that near each vertex the sum of the angles is a submultiple of 2π. The completeness condition makes note of the fact that if G were discontinuous, then we could project the Riemannian metric from U to U/G, so that U/G would become a complete metric space.

Let D be a polygon with an identification. Then there is a natural *identified polygon* D^* obtained by identifying the sides of D; i.e., there is a surjection $\pi : \bar{D} \to D^*$ where $\pi(x) = \pi(x')$ if there is a generator A with $A(x) = x'$. For $x, x' \in D^*$, we set

$$\rho^*(x, x') = \inf \sum_{i=1}^{n} \rho(z_i, z_i'),$$

where the infimum is taken over all n and over all $2n$-tuples of points of $\bar{D}$ where $\pi(z_1) = x$, $\pi(z_i') = \pi(z_{i+1})$, and $\pi(z_n') = x'$.

Our polygon D is called *complete* if

(e) for each $x \in D^*$, $\pi^{-1}(x)$ is a finite set, in which case ρ^* is a metric on D^*, and

(f) D^* is complete in this metric.

The basic property of the metric ρ^* is that

$$\rho^*(\pi(z), \pi(z')) \leqslant \rho(z, z')$$

so that $\pi : \bar{D} \to D^*$ is continuous.

We now assume that D is a complete polygon with an identification. Let z_1 be some vertex of D. There are precisely two sides of D which meet at z_1; we choose one of these and call it s_1. There is then a corresponding side s_1' and a generator $A_1 = A(s_1, s_1')$. Set $z_2 = A_1(z_1)$ and observe that there is a unique other side s_2 which has z_2 as an end point. There is a corresponding side s_2' and a generator $A_2 = A(s_2, s_2')$. Set $z_3 = A_2(z_2)$ and let s_3 be the unique other side which has z_3 as an endpoint, and so on. In this manner we get a sequence $\{z_i\}$ of vertices, a sequence $\{(s_i, s_i')\}$ of pairs of sides, and a sequence $\{A_i\}$ of generators.

All of these vertices are of course mapped into the same point of D^*, and so by condition (e), the sequence of vertices is periodic. Since each vertex lies on the boundary of precisely two sides, the sequence of pairs of sides is periodic, and so is the sequence of generators.

These sequences need not all be periodic with the same period; there is a trivial example of this. Let x be a point on the line L. Let D be one of the half planes bounded by L, let A denote reflection in L, and let s_1 and s_2 be the closed half-lines, lying on L with end point x. Then, starting at the vertex x, the sequence of vertices $\{x, x, x,...\}$ has period one, the sequence of generators $\{A, A, A,...\}$ has period one, but the sequence of pairs of sides $\{(s_1, s_1), (s_2, s_2), (s_1, s_1),...\}$ has period two.

We define the *period* to be the least positive integer m so that all three sequences are periodic with period m. The set of vertices $(z_1,..., z_m)$ is called a *cycle* of vertices.

We have already observed that the vertices in a cycle need not be distinct; in fact one easily sees that there can be repetitions only if two out of the set of generators $\{A_1,..., A_m\}$ are reflections, and then each vertex appears precisely twice in the cycle.

We are mainly interested in the isometry $B = A_m \circ \cdots \circ A_1$ which is called the *cycle transformation* at z_1. One sees at once that $B(z_1) = z_1$, and that the cycle transformation at z_i is a conjugate of the cycle transformation at z_1.

For each vertex z_i in a cycle, the sides s'_{i-1} and s_i make an angle $\alpha(z_i)$ at z_i, where $\alpha(z_i)$ is the angle measured from inside D.

The polygon D is said to satisfy the *cycle condition* if

(g) for each cycle $\{z_1, ..., z_m\}$, there is an integer ν, so that $\nu \sum_{i=1}^{m} \alpha(z_i) = 2\pi$.

Looking near z_1, we can see D and its transforms in the order:

$$D, A_1^{-1}(D), A_1^{-1} \circ A_2^{-1}(D), ..., B^{-1}(D), A_1^{-1} \circ B^{-1}(D), A_1^{-1} \circ A_2^{-1} \circ B^{-1}(D), ..., B^{-\nu}(D).$$

Condition (g) thus has the following consequences. *The cycle transformation B is orientation preserving, and $B^\nu = 1$.*

We get a relation of the form $B^\nu = 1$ for each cycle, and we call these relations the *cycle relations.*

If D is a complete polygon with an identification, and D satisfies the cycle condition, then D is called a *Poincaré polygon.*

THEOREM (Poincaré). *Let D be a Poincaré polygon. Let G be the group generated by the identifying generators. Then G is discontinuous, D is a fundamental polygon for G, and the cycle relations together with the reflection relations form a complete set of relations for G.*

3. The basic idea of the proof is as follows. We look at $\bar{D}$ together with all of its translates under G. There is a natural notion of continuation from $\bar{D}$ to $A(\bar{D})$, if A is a generator. Hence we can look at $\bar{D}$, together with all its translates, as a possibly branched, possibly bounded covering of U. The only possible ramification occurs at the vertices and their translates, the cycle condition takes care of that possibility. We next use the completeness condition to show that we can lift paths from U to this covering. Then since U is simply-connected, we get that there is actually no overlap between D and any of its translates under G.

4. Before we go on to the formal proof, there are two remarks which should be made. We need to relate this formulation of the theorem with the classical formulation, and we need to generalize the theorem to three-dimensional hyperbolic space. We take up the classical relation here, the generalization will be taken up after the formal proof.

Poincaré's classical theorem deals with a finite sided polygon D, with an identification where the identifying generators are all orientation preserving. Condition (e) is automatically satisfied, and we assume that the cycle condition (g) is satisfied. We want to find another formulation for the completeness condition.

For this section we depart from our convention and regard U as embedded in the plane; let C be the boundary of U. The sides of D which are lines or half-lines have closures which intersect C; these points of intersection are called *infinite vertices*. We separate the infinite vertices into ideal boundary points of D, so that each infinite vertex is an end point of either one or two sides of D. For each infinite vertex x_1, we can form a *chain* of vertices as follows. x_1 is an end point of the side s_1; there is a generator $A_1 = A(s_1, s_1')$; set $x_2 = A_1(x_1)$. If x_2 is an end point of two sides s_1' and s_2, then we set $A_2 = A(s_2, s_2')$, set $x_3 = A_2(x_2)$, and continue in this manner.

The above process either stops at an infinite vertex which is the endpoint of only one side, or returns to the original vertex x_1, in which case we have a *cycle* of infinite vertices. Notice that if we have a cycle, then each infinite vertex in the cycle is the end point of two sides of D. If x_1 is an infinite vertex in a cycle, then, as in the case of ordinary vertices, we can form a product of generators B called the *infinite cycle transformation*, so that $B(x_1) = x_1$.

The classical parabolic cusp condition is

(f′) For each cycle of infinite vertices, the infinite cycle transformation B is parabolic.

In the special case that we are considering, conditions (f) and (f′) are equivalent. It is easy to prove that D is complete if there are no infinite cycles. One also easily sees that it suffices to consider the case of a cycle with exactly one infinite vertex; we change normalization so that U is the upper half plane, and place the infinite vertex at infinity. We normalize further and assume that for some $a, b > 0$, the region $E = \{1 < \mathrm{Re}(z) < a, \mathrm{Im}(z) > b\}$ is contained in D. One easily verifies that if the transformation $z \to z + a$ is the generator identifying the infinite sides of $\bar{E}$, then every Cauchy sequence in $\pi(\bar{E})$ is bounded, hence convergent.

On the other hand, if the infinite sides of $\bar{E}$ are identified by $z \to az$, then we construct the following sequence. Let $z_1 = a + iab$, let $w_i = z_i - a$, and let $z_{i+1} = aw_i$. Then $\pi(z_{i+1}) = \pi(w_i)$. Let α_i be the Euclidean straight line joining z_i to w_i, and observe that the non-Euclidean length of α_i is $b^{-1}a^{1-i}$. Hence

$$\rho^*(\pi(z_i), \pi(z_{i+1})) \leqslant b^{-1}a^{1-i},$$

and so $\{\pi(z_i)\}$ is a nonconvergent Cauchy sequence.

5. We now come to the formal proof of the theorem. Let G^* be the abstract group generated by the identifying generators, and having the cycle and reflection relations as a complete set of relations. Let $\sigma : G^* \to G$ be the natural homomorphism.

For (z, g) and (z', g') in $\bar{D} \times G^*$, we define $(z, g) \sim (z', g')$ if $z \in \partial D$ and there is an identifying generator A so that $Az = a'$ and $g' \circ A = g$. This relation need not be transitive, but it generates an equivalence relation. Let X be $\bar{D} \times G^*$ factored by this equivalence relation. We endow X with a topology in the usual manner; i.e., a subset of X is open if and only if its inverse image in $\bar{D} \times G^*$ is open, where G^* has the discrete topology. One easily observes that X is connected and Hausdorff.

There is a natural map $p : X \to U$ given by $p(z, g) = \sigma(g)(z)$. There is also a natural map $s : X \to D^*$ obtained by projecting onto the first factor followed by the map π.

These maps p and s have certain special properties. For each point $x \in X$, there is a neighborhood V, so that $p \mid V$ is a homeomorphism. This neighborhood V also has the property that $s \circ p^{-1}$ is a contraction on $p(V)$. We will prove these statements below, and using these two properties, the completeness of D^*, and the simple-connectivity of U, we will show that p is a homeomorphism.

We need to construct a special system of neighborhoods for points of D^*. If $z \in D$, let δ be some number less than half the distance from z to ∂D, let V_0 be the open disc of radius δ about z, and let $V_\delta = \pi(V_0)$.

If $z \in \bar{D}$ lies on a side s, but z is not a vertex, there is a generator A, and a side s', with $A(s) = s'$. We set $z' = A(z)$, and recall condition (c) which asserts that $z = z'$ if and only if $s = s'$. Let δ_0 be the distance from z to $\partial D - s$, and let δ_0' be the distance from z' to $\partial D - s'$. Let $\delta \leqslant 1/4 \min(\delta_0, \delta_0')$, let $V_0(V_0')$ be the open disc of radius δ about $z(z')$, and let $V_\delta = \pi(V_0 \cap \bar{D}) \cup \pi(V_0' \cap \bar{D})$.

If z_1 is a vertex, then let $z_1, z_2, \ldots, z_n$ be the cycle of vertices containing z_1. For each vertex z_i, let s_i, s_{i-1}' be the sides which intersect at z_i. Let δ_i be the distance from z_i to $\partial D - \{s_i \cup s_{i-1}'\}$. We choose $\delta \leqslant 1/4 \min(\delta_1, \ldots, \delta_n)$; let V_i be the disc about z_i of radius δ, and let $V_\delta = \bigcup_{i=1}^n \pi(\bar{D} \cap V_i)$.

The point of these special neighborhoods is the following. For each point $z \in D^*$, and special neighborhood V_δ of z, V_δ is a metric ball of radius δ about z. Further $\pi^{-1}(V_\delta)$ is a finite union of disjoint sets $\pi^{-1}(V_\delta) = V_1 \cup \cdots \cup V_n$ with the property that the distance between any two of these sets is not less than δ.

If we now let $\tilde{V}_\delta$ be some connected component of $s^{-1}(V_\delta)$, then the identification and cycle conditions imply that $p \mid \tilde{V}_\delta$ is a homeomorphism. Furthermore, the inequalities for δ have been chosen so that if $x, x' \in \tilde{V}_\delta$, then

$$\rho^*(s(x), s(x')) \leqslant \rho(p(x), p(x')). \tag{1}$$

This inequality is of course an equality except near the vertices where the integer ν in condition (g) is greater than one.

Using this inequality, we observe that we can lift rectifiable paths. Let us assume that the origin 0 is in D, and that $w(t)$, $0 \leqslant t \leqslant 1$, is a rectifiable path in U, starting at 0. Since $p : X \to U$ is a local homeomorphism, we can assume that there is a path $\tilde{w}(t)$, $0 \leqslant t < 1$, in X, covering $w(t)$ and starting at (0, 1). Let t_n be some sequence of points of the unit interval with $t_n \to 1$, and assume that the t_n are sufficiently close together so that for every n there is a special neighborhood V with $s \circ \tilde{w}(t_n)$ and $s \circ \tilde{w}(t_{n+1})$ both lying in V.

Now for any n and m

$$\rho^*(s \circ \tilde{w}(t_m), s \circ \tilde{w}(t_n)) \leqslant \sum_{i=m}^{n-1} \rho(w(t_m), w(t_{m+1})), \tag{2}$$

where we have used (1) and the fact that $w(t_i) = p \circ \tilde{w}(t_i)$.

We now use the fact that w is rectifiable together with the completeness of D^*, to conclude that $s \circ \tilde{w}(t_n)$ has a limit x. We let V_δ be a special neighborhood about x, and observe that there is a connected component $\tilde{V}_\delta$ of $s^{-1}(V_\delta)$ so that $\tilde{w}(t_n) \in \tilde{V}_\delta$ for n sufficiently large. Since $p \mid \tilde{V}_\delta$ is a homeomorphism, we can complete the lifting of w. We have shown that every rectifiable path in U, starting at 0, can be lifted to X. In particular, p is surjective.

It is important for the generalization to stop at this point and observe that the proof above uses only inequality (1), the completeness of D^*, and the fact that p is a local homeomorphism.

For each positive t, let B_t be the open ball of radius t about 0. Let

$$\tau = \sup\{t \mid p^{-1} \text{ is well defined on } B_t\}.$$

Suppose τ were finite. Then for each z on the boundary of B_τ, we could lift the straight line from 0 to z, and hence p^{-1} would be well defined on $\bar{B}_\tau$. Trivially, if p is injective on a compact set, it is injective in a neighborhood of the compact set. We conclude that p^{-1} is well defined in a neighborhood of $\bar{B}_\tau$, contradicting the definition of τ.

We remark that X is connected, and conclude that p is a homeomorphism. This completes the proof of the theorem.

6. There is also a classical generalization of Poincaré's theorem using a similar proof, to three-dimensional hyperbolic space, where one gets a Kleinian rather than a Fuchsian group. The theorem can be generalized further to n-dimensional hyperbolic space, but the hypotheses become difficult to check in any given case.

We now specifically assume that U is hyperbolic 3-space.

A *polyhedron* D is an open connected subset of U where ∂D is the union of countably many *sides* $\{s_i\}$ as follows. Each side s_i is a subset of a plane L_i, and as a subset of the hyperbolic plane L_i, s_i is the closure of a polygon. The sides of the polygon s_i are called *edges* of D, and the (finite) end points of these edges are called *vertices* of D. We require that any compact set meets only finitely many sides, edges, and vertices. Further, for each edge e_j there are exactly two sides s_i and s_j so that $s_i \cap s_j = e_j$. Any two sides are either disjoint, intersect in a common edge, or intersect in a common vertex. An edge is either a subset of a side, meets the side in a common vertex, or is disjoint from the side. Two edges are disjoint or meet in a common vertex. Finally, for each $x \in \partial D$ and each sufficiently small δ, the ball about x of radius δ intersects D in a connected set.

An *identification* on a polyhedron is exactly the same as an identification on a polygon; i.e., it is a pairing of the sides via isometries, satisfying conditions (a)–(d). Likewise, we define the *identified polyhedron* D^* with metric ρ^*, and the projection $\pi : \bar{D} \to D^*$ exactly as in the two-dimensional case. Having done this, the completeness conditions (e) and (f) make sense.

Having moved up one dimension, it is now the edges that come in cycles. For each edge e_1, exactly as before, we form the cycle of edges $e_1, \ldots, e_m$, we get the cycle transformation B, and for each edge e_i we have the angle $\alpha(e_i)$ formed by the two sides that meet at e_i. Except for minor notation, condition (g) now makes sense for polyhedra.

We would like to know that the cycle relation $B^\nu = 1$ holds. Unfortunately, the two-dimensional analysis only gives us that B keeps e_1 invariant, preserves orientation in the plane normal to e_1, and B^ν is the identity in the normal plane.

In order to guarantee the validity of the cycle relations, we impose two additional conditions.

(h) For each edge e_1, the cycle transformation B at e_1 preserves orientation.

(i) If the edge e_1 has no finite end point, then the cycle transformation at e_1 is the identity on e_1.

Condition (h) is of course vacuous if all the generators preserve orientation; I do not know if condition (h) is independent of the others.

Condition (i) is necessary, as the following example demonstrates. Let D be bounded by four sides $s_1, \ldots, s_4$, where s_1 and s_2 are orthogonal, s_3 and s_4 are orthogonal, and they are otherwise disjoint. Choose generators A_1, A_2, so that $A_1(s_1) = s_3$, $A_2(s_4) = s_2$. Observe that there is a one-parameter freedom in the choice of both A_1 and A_2. Hence in general $A_2 \circ A_1$ is not elliptic.

We have to show that conditions (h) and (i) are sufficient to guarantee that $B^\nu = 1$. Condition (h) says that B preserves orientation. We already know that B preserves orientation in the plane normal to e_1, hence B preserves orientation on e_1. If e_1 has finite length, then B is the identity on e_1. If e_1 has a single finite endpoint x, then $B(x) = x$, and hence e_1 lies on a fixed line for B. Using condition (i) for the last case, we see that B keeps e_1 pointwise fixed, and since B^ν is the identity in the normal plane, we conclude that $B^\nu = 1$.

Now that we have established the validity of the cycle relations, we can state the polyhedron theorem.

THEOREM (Poincaré). *Let D be a polyhedron satisfying conditions* (a)–(i). *Let G be the group generated by the identifications of the sides. Then G is discontinuous, D is a fundamental polyhedron for G, and the cycle relations together with the reflection relations form a complete set of relations for G.*

The proof of this theorem is essentially the same as the two dimensional case. We again form the group G^* and the space X, where X is $\bar{D} \times G^*$ factored by the same equivalence relation. We also have the maps $p : X \to U$, and $s : X \to D^*$.

For each $x \in D^*$, we construct the family V_δ of special neighborhoods, as in the preceding case so that inequality (1) holds.

We have to check that p is a homeomorphism on each connected component $\tilde{V}_\delta$ of $s^{-1}(V_\delta)$. We write $x = \pi(y)$, and observe the following. If y is an interior point of D, then p is trivially a homeomorphism in a neighborhood of y. If y is an interior point of a side, then we use the identification conditions. If y is an interior point of an edge, then we have set up conditions (g), (h), and (i) precisely so that p is a homeomorphism near $s^{-1} \circ \pi(y)$.

The remaining case, which is new in this dimension, is that y is a vertex. We observe trivially that p is a local homeomorphism on the boundary of $\tilde{V}_\delta$, and (1) holds. Hence we can lift rectifiable paths from ∂V_δ to $\partial \tilde{V}_\delta$. The "greatest schlicht disc" argument that we used before shows that every point of ∂V_δ has a neighborhood, homeomorphic to each connected component of its preimage under p. Since ∂V_δ is a 2-sphere, we conclude that $p \mid \partial \tilde{V}_\delta$ is a homeomorphism.

The above argument holds for every sufficiently small δ, and so $p \mid \tilde{V}_\delta$ is a homeomorphism.

The remainder of the proof of the theorem is identical with that given in the two-dimensional case.

7. We conclude this discussion with a geometric criterion for the completeness condition in the case that the polyhedron has finitely many sides, and all the generators preserve orientation.

We again let C be the boundary of U, where we regard U as the unit ball in real 3-space.

It is trivial that D^* is complete if $\bar{D}$ is compact. There is an obvious extension of this to the case that D^* can be compactified by adjoining points of C to D.

To make the above remark explicit, we observe that any two planes in U have disjoint closures, intersect in a line, or have a point of tangency on C. Each of the sides s_i, $i = 1,..., n$, lies in a plane L_i, and, if each pair of planes L_i, L_j, either have disjoint closures or intersect in a line, then D^* is complete. To see this we simply observe that near C, the closure of D is a finite union of finite-sided polygons crossed with an interval. The collection of finite-sided polygons, when identified, is obviously complete.

In the above argument we did not need L_i and L_j not to be tangent; we only needed to know that there is no point of tangency which is also a boundary point of D. We call such a point a *tangency vertex.* Precisely as in the two-dimensional case we observe that the tangency vertices come in chains or cycles, and for each cycle starting at x_1, we have the *tangency vertex transformation* B which keeps x_1 fixed. We repeat the two-dimensional argument and prove that D^* *is complete if and only if each tangency vertex transformation is parabolic.*

We conclude by remarking that this last condition is a restatement of condition (f) in the classical case only; it does not in any case replace condition (i).

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