Notations:

$T: X \to X$ a homeomorphism of a compact metric space

$h_{cover}(T) = \sup\limits_{\mathcal{U} \text{ a cover of } X} \lim\limits_{n\to\infty} \frac{1}{n} \log N(\mathcal{U}_0^{n-1})$, where

$N(\mathcal{V})$ = minimal cardinality of a subcover of $\mathcal{V}$

$\mathcal{U}_0^{n-1} = \bigvee\limits_{i=0}^{n-1} T^{-i}(\mathcal{U})$.

$d_n(x,y) = \max\limits_{i=0,\dots,n-1} d(T^i x, T^i y)$ - Bowen metric on X.

$S_{sep}(n,\varepsilon)$ = maximal cardinality for an ε-separated set for (X, d_n).

$S_{span}(n,\varepsilon)$ = minimal cardinality of an ε-spanning set for (X, d_n).

$h_{span}(T) = \lim\limits_{\varepsilon\to 0+} \limsup\limits_{n\to\infty} \frac{1}{n} \log S_{span}(n,\varepsilon)$.

$h_{sep}(T) = \lim\limits_{\varepsilon\to 0+} \limsup\limits_{n\to\infty} \frac{1}{n} \log S_{sep}(n,\varepsilon)$.

<u>Thm</u> $h_{cover}(T) = h_{sep}(T) = h_{span}(T) = \lim\limits_{\varepsilon\to 0+} \liminf\limits_{n\to\infty} \frac{1}{n} \log S_{sep}(n,\varepsilon)$

$= \lim\limits_{\varepsilon\to 0+} \liminf\limits_{n\to\infty} \frac{1}{n} \log S_{span}(n,\varepsilon)$.

<u>Pf</u> We proved in the lecture $S_{span}(n,\epsilon) \le S_{sep}(n,\varepsilon) \le S_{span}(n,\varepsilon/2)$ and this easily implies $h_{sep}(T) = h_{span}(T)$.

Let $\mathcal{U}^{(\varepsilon)}$ be the cover of X by all balls of radius 2ε, and let $\mathcal{V}^{(\varepsilon)}$ be some (any) cover by balls of radius $\varepsilon/2$. The sets $\left\{\bigcap\limits_{k=0}^{n-1} T^{-k}(B(T^k x, 2\varepsilon)) : x \in F\right\} \subset \mathcal{U}_0^{n-1}$ cover X, where F is a spanning set for (X, d_n), by def. of d_n, ~~implies~~ $|F| = S_{span}(n,\varepsilon)$. This implies $N(\mathcal{U}_0^{n-1}) \le S_{span}(n,\varepsilon)$.

Now if $F' \subset X$, $|F'| = S_{sep}(n,\varepsilon)$, and $x, y \in F'$, $x \ne y$, then x, y can't belong to the same element of $\mathcal{V}_0^{n-1}$

This implies $S_{sep}(n,\varepsilon) \le N(\mathcal{V}_0^{n-1})$, and all together

$$N(\mathcal{U}_0^{n-1}) \le S_{span}(n,\varepsilon) \le S_{sep}(n,\varepsilon) \le N(\mathcal{V}_0^{n-1})$$

Taking logs, etc., this gives

$$h_{cover}(T,\mathcal{U}) \le \liminf_{n\to\infty} \frac{1}{n} \log S_{span}(n,\varepsilon) \le$$

$$\le \liminf_{n\to\infty} \frac{1}{n} \log S_{sep}(n,\varepsilon) \le$$

$$\le \limsup_{n\to\infty} \frac{1}{n} \log S_{sep}(n,\varepsilon) \le h_{cover}(T,\mathcal{V})$$

Letting $\varepsilon \to 0$ gives

$$h_{cover}(T,\mathcal{U}^{(\varepsilon)}) \le \lim_{\varepsilon\to 0+} \liminf_{n\to\infty} \frac{1}{n} \log S_{span}(n,\varepsilon)$$

(*) $$\le \lim_{\varepsilon\to 0+} \liminf_{n\to\infty} \frac{1}{n} \log S_{sep}(n,\varepsilon) \le h_{sep}(T)$$

$$\le h_{cover}(T, \mathcal{V}^{(\varepsilon)})$$

We showed that for any sequence $\mathcal{W}^{(n)}$ of covers with $\text{diameter}(\mathcal{W}^{(n)}) \xrightarrow[n\to\infty]{} 0$, we have $h_{cover}(T,\mathcal{W}^{(n)}) \xrightarrow[n\to\infty]{} h_{cover}(T)$

Therefore both sides of (*) tend to $h_{cover}(T)$ as $\varepsilon \to 0$.

This implies the theorem.