Let G be an lcsc group, X an lcsc space, $G \curvearrowright X$ continuously and transitively. Suppose $\nu \ll \mu$, where μ and ν are two regular Radon G-invariant Borel measures on X, with $\mu \neq 0$, $\nu \neq 0$. Then there is $\alpha > 0$ s.t. $\nu = \alpha\mu$.

<u>Proof</u>: Let $\varphi = \frac{d\nu}{d\mu}$, that is $\varphi \in L^1(\mu)$ and for all $f \in L^1(\mu)$, $\int f \, d\nu = \int f \varphi \, d\mu$. From the Radon-Nikodym theorem, which can be applied since μ, ν are σ-finite, such a φ exists and is unique up to μ-nullsets.

Let $g_0 \in G$. We show that there is $A = A(g_0) \in \mathcal{B}$, with $\mu(A^c) = 0$, s.t. $\varphi(g_0 x) = \varphi(x)$ for all $x \in A$.

Indeed, for all f,

$$\int f \, d\nu = \int f \circ g_0^{-1} \, d\nu = \int (f \circ g_0^{-1}) \cdot \varphi \, d\mu = \int f \cdot \varphi \circ g_0 \, d\mu$$

ν is g_0-invariant property of φ μ is g_0-invt.

So $\varphi \circ g_0$ has the defining property of $\frac{d\nu}{d\mu}$, so $\varphi \circ g_0 = \varphi$ on a set of full μ-measure. This proves the claim.

We now use Fubini for $G \times X$: let m_G be Haar on G

$$0 = \int_G 0 \, dm_G = \int_G \int_X |\varphi(g_0 x) - \varphi(x)| \, d\mu(x) \, dm_G(g_0)$$

prev. claim

$$= \int_X \int_G |\varphi(g_0 x) - \varphi(x)| \, dm_G(g_0) \, d\mu(x)$$

Fubini, (X, μ) σ-finite

$\Rightarrow \exists X_0 \subset X$, $\mu(X \setminus X_0) = 0$ s.t. $\forall x \in X_0$,

$$\int_G |\varphi(g_0 x) - \varphi(x)| \, dm_G(g_0) = 0$$

$\Rightarrow \exists X_0 \in \mathcal{B}, \mu(X \setminus X_0) = 0$ s.t.

(*) $\forall x \in X_0$, for m_G-a.e. $g_0 \in G$, $\varphi(g_0 x) = \varphi(x)$.

Let $x_0 \in X$ and let

$C = \{x \in X : \varphi(x) \neq \varphi(x_0)\}$

If (by contradiction) $\mu(C) > 0$ then there is $x_1 \in X_0 \cap C$, so (*) holds for x_1.

There is $g_0 \in G$ s.t. $g_0 x_0 = x_1$ by transitivity of the G-action. Setting

$B_0 = \{g \in G : \varphi(g x_0) = \varphi(x_0)\}$, $B_1 = \{g \in G : \varphi(g x_1) = \varphi(x_1)\}$

we get $B_0 g_0 \cap B_1 = \emptyset$. This contradicts that m_G is left g_0-invariant and $m_G(B_0^c) = m_G(B_1^c) = 0$. We have a contradiction proving $\mu(C) = 0$. Set $\alpha = \varphi(x_0)$, then $\varphi = \alpha$ on a subset of X of full measure.

Hence $\nu = \alpha\mu$.