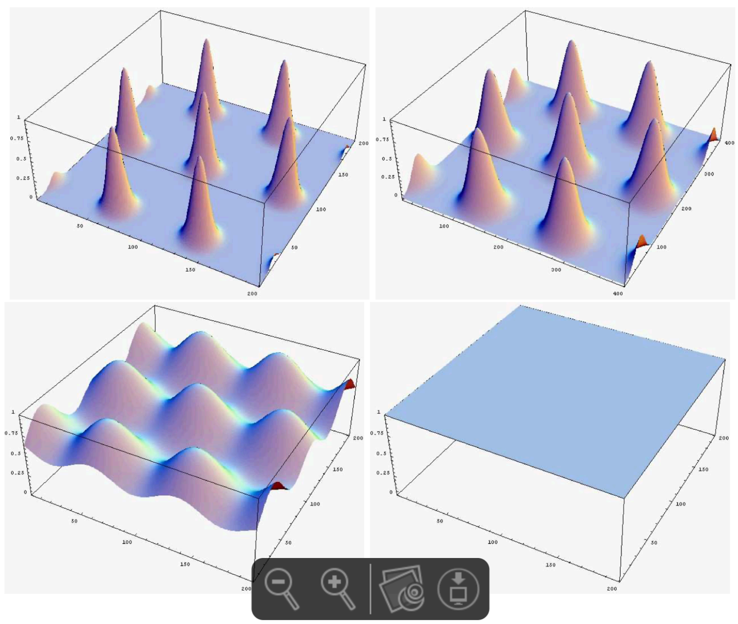


Geometry of Numbers
Lecture 13

Recall: Want to simulate Haar-Siegel measure $m_{\mathcal{X}_n}$ on $\mathcal{X}_n$. What does this mean?

Want ν_j on $\mathcal{X}_n$ s.t. $\nu_j \xrightarrow[j\to\infty]{} m_{\mathcal{X}_n}$ weak-*,

Radon measures

i.e. $\forall f \in C_c(\mathcal{X}_n)$, $\int_{\mathcal{X}_n} f \, d\nu_j \xrightarrow[j\to\infty]{} \int_{\mathcal{X}_n} f \, dm_{\mathcal{X}_n}$

(ν_j could be finitely supported measures,
f is called a test function).

Sometimes also want to understand rate of convergence.

Example $X = \mathbb{R}$

$$\nu_N = \frac{1}{N} \sum_{x \in \{-N, -N+\frac{1}{N}, \dots, N\}} \delta_x ,$$

$$\int f \, d\nu_N = \frac{1}{N} \sum_{i=0}^{2N^2} f\left(-N + \frac{i}{N}\right).$$

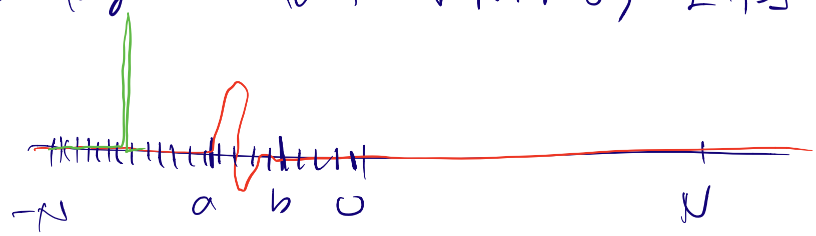
scale by #pts
in a unit interval

Claim: $\nu_N \longrightarrow$ Lebesgue on $\mathbb{R}$.

Pf Suppose $f \in C_c(\mathbb{R})$, $\operatorname{supp} f \subset [a,b]$

$$\int_{\mathbb{R}} f \, d\mathrm{Vol} = \int_a^b f(x)\,dx$$

Take N_0 large so that $\forall N \geq N_0$, $[a,b] \subset [-N,N]$



$$\int f \nu_N = \frac{1}{N} \sum_{i=0}^{2N^2} f\left(-N + \frac{i}{N}\right)$$

= Riemann sum for f corresponding to

the partition $\xrightarrow[N\to\infty]{} \int_a^b f(x)\,dx$

$$-N = x_0 < x_1 = x_0 + \frac{1}{N} < \dots < x_{2N^2} = N$$

by "def" of Riemann integral

Suppose test function f is Lipschitz, i.e. $\exists C > 0$

$\forall x, y \quad |f(x) - f(y)| \leq C|x-y|$.

Then on each intervals $x \in [y, y + \frac{1}{N}]$ in partition

$$|f(x) - f(y)| \leq C|x-y| \leq \frac{C}{N}$$

$$\left| \int_y^{y+\frac{1}{N}} f(x)\,dx - \frac{1}{N} f(y) \right| \le \int_y^{y+\frac{1}{N}} |f(x) - f(y)|\, dx$$

$$\le \frac{C}{N} \cdot \frac{1}{N} = \frac{C}{N^2}$$

if $[y, y+\frac{1}{N}] \cap \operatorname{supp} f = \emptyset$ error is zero, otherwise bdd by $\frac{C}{N^2}$

$$\left| \int_a^b f(x)\,dx - \int f \, d\nu_N \right| \le \frac{C}{N^2} \cdot N(b-a) \le \frac{C(b-a)}{N}$$

Simulating Haar-Siegel measure.

Use "Hecke-correspondence":

Let $L_0 \in \mathcal{X}_n$, p be a prime.

$L' \in \mathcal{X}_n$ is called a <u>p-Hecke friend</u> of L_0

if $\exists L_1 \subset L_0$, s.t. $[L_0 : L_1] = p$, s.t. $L' = p^{\frac{1}{n}} L_1$

Define $\nu_p = \frac{1}{N_p} \sum_{\substack{L' \text{ a p-Hecke} \\ \text{friend of } L_0}} \delta_{L'}$

where $N_p = \#\{\text{p-Hecke friends of } L_0\}$.

Proving $\nu_p \xrightarrow[p\to\infty]{} m_{\mathcal{X}_n}$ weak-* is beyond the scope of this course.

We will

$$\int_{\mathcal{X}_n} f\, d\nu_p \xrightarrow[p\to\infty]{} \int_{\mathcal{X}_n} f\, dm_{\mathcal{X}_n} \quad (*)$$

for test functions of the form

$$f(L) = \sum_{v \in L \setminus 0} \varphi(v), \quad \text{where } \varphi \in C_c(\mathbb{R}^n).$$

<u>Proof of (*) for such f:</u> $\int_{\mathcal{X}_n} f\, d\nu_p$

$$= \frac{1}{N_p} \sum_{[L_0:L_1]=p} f(p^{-\frac{1}{n}} L_1) = \frac{1}{N_p} \sum_{[L_0:L_1]=p} \sum_{v \in p^{-\frac{1}{n}} L_1 \setminus 0} \varphi(v)$$

$$= \frac{1}{N_p} \sum_{[L_0:L_1]=p} \left(\sum_{v \in p^{1-\frac{1}{n}} L_0 \setminus 0} \varphi(v) + \sum_{v \in p^{-\frac{1}{n}}(L_1 \setminus pL_0)} \varphi(v) \right)$$

$pL_0 \subset L_1 \subset L_0$

first summand does not depend on L_1

$$= \underbrace{\sum_{v \in p^{1-\frac{1}{n}} L_0 \setminus 0} \varphi(v)}_{A} + \underbrace{\frac{1}{N_p} \sum_{[L_0:L_1]=p} \sum_{v \in p^{-\frac{1}{n}}(L_1 \setminus pL_0)} \varphi(v)}_{B}$$

If p is large enough, any vector in $p^{1-\frac{1}{n}} L_0 \setminus 0$ is outside $\operatorname{supp} \varphi$. So $A \xrightarrow[p\to\infty]{} 0$.

<u>Claim:</u> In the sum B, every $v \in L_0 \setminus pL_0$ is chosen with the same frequency, $\frac{p^{n-1}-1}{p^n-1}$.

Assuming the claim,

$$B = \frac{p^{n-1}-1}{p^n-1} \sum_{v \in p^{-\frac{1}{n}}(L_0 \setminus pL_0)} \varphi(v)$$

$$= \frac{1}{p} \sum_{v \in p^{-\frac{1}{n}} L_0} \varphi(v) + o(1) \xrightarrow[p\to\infty]{} \int_{\mathbb{R}^n} \varphi \, d\mathrm{Vol}$$

$\frac{p^{n-1}-1}{p^n-1} = \frac{1}{p} + o(1)$ as $p \to \infty$

by def of Riemann

integral $\qquad\qquad \| (SSF)$

added back to the sum $\qquad \int f \, dm_{\mathcal{X}_n}$

$v \in p^{1-\frac{1}{n}} L_0$, all but one of them $(v=0)$ $\quad \mathcal{X}_n$

don't belong to supp φ for large p and don't affect B.

<u>Pf of Claim</u> Consider $L_0/pL_0 = \{0,1,\dots,p-1\}^n$

$= \mathbb{F}_p^n$, $\mathbb{F}_p = \{0,1,\dots,p-1\}$ the field

with p elements.

Any subgroup of L_0/pL_0 is actually a vector

subspace. Projection of L_1 to $\mathbb{F}_p^n$ is a

subspace of dimension $n-1$.

Each v is counted according to the number

of $n-1$ dimensional subspaces containing it.

$SL_n(\mathbb{F}_p)$ permutes the vectors and subspaces,

acts transitively on vectors, so every vector

belongs to the same # of $n-1$-dim.

subspaces. So every v is sampled the same number of times in $\mathcal{B}$. The number of times is

$$\frac{\#\begin{pmatrix}\text{nonzero vectors in} \\ \text{an } n-1 \text{ dim subspace}\end{pmatrix}}{\#\begin{pmatrix}\text{nonzero} \\ \text{vectors in } \mathbb{F}_p^n\end{pmatrix}} = \frac{p^{n-1}-1}{p^n-1}$$

Another way of simulating $m_{\mathcal{X}_n}$:

"equidistribution of horospheres".

Let $g_t = \begin{pmatrix} e^t & & 0 \\ & \ddots & \\ 0 & & e^t \\ & & & e^{-(n-1)t}\end{pmatrix} \in SL_n(\mathbb{R})$

$$u_{\underline{a}} = \begin{pmatrix} 1 & & 0 & a_1 \\ & \ddots & & \vdots \\ & & \ddots & a_{n-1} \\ 0 & & & 1 \end{pmatrix} \qquad \underline{a} \in [0,1)^{n-1} \qquad \underline{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_{n-1}\end{pmatrix}$$

Define ν_t on $\mathcal{X}_n$ by

$$\int_{\mathcal{X}_n} f \, d\nu_t = \int_{[0,1)^{n-1}} f(g_t u_{\underline{a}} L_0) \, d\mathrm{Vol}(\underline{a}).$$

$L_0 = \mathbb{Z}^n$.

Thm (horospherical equidistribution)

$$\nu_t \xrightarrow[t\to\infty]{} m_{\mathcal{X}_n} \quad \text{weak-* top.}$$

(proved in hom. dynamics course last year)

see ex. sheet.

Hecke operators

$$\mathbb{T}_p : C_c(\mathcal{X}_n) \circlearrowleft$$

$$\mathbb{T}_p : L^2(\mathcal{X}_n, m_{\mathcal{X}_n}) \circlearrowleft$$

$$(\mathbb{T}_p \psi)(L_0) = \sum_{[L_0:L_1]=p} \psi(p^{-1/n} L_1)$$

Covering volume and covering radius

If L is a lattice

euclidean ball

$$\operatorname{covrad}(L) = \inf\{r>0 : L + B(0,r) = \mathbb{R}^n\}$$

$$= \max_{y\in\mathbb{R}^n} \operatorname{dist}(y, L)$$

covering volume $\Theta(L) = \inf\left\{ \mathrm{Vol}(B) : \begin{array}{l} B \text{ euclidean} \\ \text{ball,} \\ B+L=\mathbb{R}^n \end{array} \right\}$.

$L^* = \{u \in \mathbb{R}^n : \forall x \in L \ \langle x,u\rangle \in \mathbb{Z}\}$

"dual lattice of L".

Banaszczyk transference theorem ('93)

Thm $\forall n \ \forall L$ (lattice in $\mathbb{R}^n$):

(i) $\lambda_1(L) \cdot \mathrm{covrad}(L^*) \leq n$

(ii) $\exists c$ s.t. for all L, $\forall i$, $\forall n$

$$\lambda_i(L) \cdot \lambda_{n+1-i}(L^*) \leq cn.$$

These results are sharp up to the constants.

$\exists L_n \in \mathcal{X}_n$, and $c' > 0$ s.t.

$$\lambda_1(L_n) \cdot \mathrm{covrad}(L_n^*) \geq c' n.$$

Crash course on Fourier series and Fourier transform

(We won't strive for full generality).

(A) Fourier series on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

Let f be a C^∞ function on $\mathbb{T}^n$

$\left(\Longleftrightarrow f : \mathbb{R}^n \to \mathbb{C}\right.$, which is $\left.\mathbb{Z}^n\text{-periodic}\right)$.
C^∞

standard inner product

For $y \in \mathbb{Z}^n$, $e(y \cdot x) \overset{\text{notation}}{=} e^{2\pi i \langle x, y\rangle}$

Define $\hat{f}(y) = \int_{\mathbb{T}^n} f(x)\, e(-x \cdot y)\, dx$

not to be confused with $\hat{f}$ in SSF

$\mathbb{T}^n = [0,1)^n$
$dx = d\mathrm{Vol}\, \mathbb{1}_{[0,1)^n}$.

$$f(x) = \sum_{y \in \mathbb{Z}^n} \hat{f}(y)\, e(y \cdot x).$$

(B) For a general lattice/torus:

Let $L \subset \mathbb{R}^n$ a lattice, $\mathbb{T}_L = \mathbb{R}^n/L$

$f : \mathbb{T}_L \longrightarrow \mathbb{C}$ $\quad C^\infty$.

$$\hat{f}(y) = \int_{\mathbb{T}_L} f(x)\, e(-y\cdot x)\, dx \qquad \text{for } y \in L^*$$

Haar probability measure on $\mathbb{T}_L$ $\Longleftrightarrow$ $\dfrac{d\mathrm{Vol}}{\mathrm{covol}(L)}$ restricted to a fund. domain for L

Then
$$f(x) = \sum_{y \in L^*} \hat{f}(y)\, e(y\cdot x).$$

(C) Fourier transform

For $f \in C^\infty(\mathbb{R}^n)$ and Schwarz function

(every derivative of f of every order decays subpolynomially on $\mathbb{R}^n$ as $\|\cdot\| \to \infty$).

$$\hat{f}(y) = \int_{\mathbb{R}^n} f(x)\, e(-y\cdot x)\, dx$$

dVol on $\mathbb{R}^n$

$y \in \mathbb{R}^n$.

Then $f(x) = \int_{\mathbb{R}^n} \hat{f}(y)\, e(y \cdot x)\, dy$.

<u>An easy formula</u>: $\forall \lambda > 0$, define

$h(x) = \lambda^n f(\lambda x)$, then

$$\hat{h}(y) = \hat{f}\left(\frac{y}{\lambda}\right)$$

(proof: compute).

<u>An example</u>: $\rho(x) = e^{-\pi \|x\|^2}$

(Gaussian). (ρ is a Schwarz function)

satisfies $\hat{\rho}(y) = \rho(y) \quad \forall y \in \mathbb{R}^n$.

By previous formula, if we define

$\rho_s(x) = e^{-\pi \|\frac{x}{s}\|^2}$, $\hat{\rho}_s(y) = s^n \rho_{\frac{1}{s}}(y)$.

D Poisson summation formula (PSF)

Given $f \in C^\infty(\mathbb{R}^n)$ (Schwarz function)

Given a lattice L, define the periodization of f by $f_L(x) = \sum_{v \in L+x} f(v)$

f_L is L periodic. Then

$f_L(x) = \sum_{y \in L^*} \hat{f}(y)\, e(y \cdot x)$ if $L \in \mathcal{X}_n$

$f_L(x) = \operatorname{covol}(L^*) \sum_{y \in L^*} \hat{f}(y)\, e(y \cdot x)$ for general L.

<u>Pf:</u> Let f_L be the periodization of f, we want to compute $\widehat{f_L}(y)$, for $y \in L^*$.

$$\widehat{f_L}(y) \overset{\text{def of }\wedge}{=} \int_{\mathbb{T}_L} f_L(x)\, e(-y \cdot x)\, dx$$

$$= \int_{\mathbb{T}_L} \left(\sum_{z \in L} f(x+z) \right) e(-y \cdot x)\, dx$$

def of f_L $\displaystyle = \sum_{z \in L} \int_{\Pi_L} f(x+z)\, e(-y\cdot x)\, dx$

$y \in L^*$, $z \in L$ $\displaystyle = \sum_{z \in L} \int_{\Pi_L} f(x+z)\, e(-y\cdot(x+z))\, dx$

$\displaystyle = \sum_{z \in L} \frac{1}{\operatorname{covol}(L)} \int_{\Omega_L} f(x+z)\, e(-y(x+z))\, d\mathrm{Vol}(x)$

Ω_L a fund. domain for L $\displaystyle = \frac{1}{\operatorname{covol}(L)} \int_{\mathbb{R}^n} f(x)\, e(-y\cdot x)\, d\mathrm{Vol}(x)$

$= \operatorname{covol}(L^*)\, \hat{f}(y)$

Now plug into Fourier series for f_L.

<u>Notation</u> For a countable set $\Lambda \subset \mathbb{R}^n$,

set $\displaystyle f(\Lambda) \overset{\text{def}}{=} \sum_{x \in \Lambda} f(x)$

Then PSF becomes

$$f(x+L) = f_L(x) = \mathrm{covol}(L^*) \sum_{y \in L^*} \hat{f}(y)\, e(y \cdot x)$$

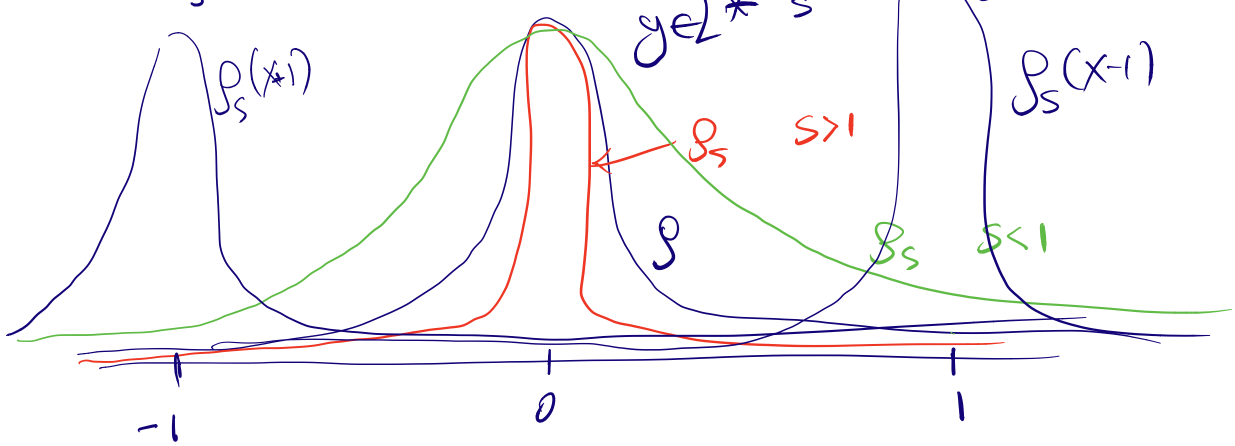
setting $x = 0$:

$$f(L) = \mathrm{covol}(L^*)\, \hat{f}(L^*).$$

Using $\rho_s(x) = e^{-\pi \| \frac{x}{s} \|^2}$, get:

(1) $\rho_s(L) = s^n \mathrm{covol}(L^*)\, \rho_{\frac{1}{s}}(L^*)$

(2) $\rho_s(L+x) = s^n \mathrm{covol}(L^*) \sum_{y \in L^*} \rho_{\frac{1}{s}}(y)\, e(y \cdot x)$.



<u>Lemma 1</u> For all L (a lattice in $\mathbb{R}^n$), all s ($s>0$) for all x ($x \in \mathbb{R}^n$)

$$\rho_s(L+x) \leq \rho_s(L).$$

Pf: $\rho_s(L+x) \in \mathbb{R}$. $\alpha \in \text{covol}(L^*)$

$$\frac{\rho_s(L+x)}{\alpha} \underset{(2)}{=} \text{Re}\left(s^n \sum_{y \in L^*} \rho_{\frac{1}{s}}(y)\, e(y \cdot x)\right)$$

$$= s^n \sum_{y \in L^*} \rho_{\frac{1}{s}}(y)\, \text{Re}(e(y \cdot x))$$

$s \in \mathbb{R}$

$\rho_{\frac{1}{s}}(y) \in \mathbb{R}$

$$\le s^n \sum_{y \in L^*} \rho_{\frac{1}{s}}(y) \underset{(1)}{=} \frac{\rho_s(L)}{\alpha}$$

$\rho_{\frac{1}{s}}(y) > 0$

Multiply through by α to get Lemma.

Lemma 2 For $s \ge 1$. For all L, all $x \in \mathbb{R}^n$,

$$\rho_s(L+x) \le s^n \rho(L).$$

Pf By Lemma 1, $\rho_s(L+x) \le \rho_s(L)$

So need to show $\rho_s(L) \le s^n \rho(L)$.

By (1), setting $\alpha = \text{covol}(L^*)$

$$\rho_s(L) \underset{(1)}{=} \alpha s^n \rho_{\frac{1}{s}}(L^*) = \alpha s^n \sum_{y \in L^*} \rho_{\frac{1}{s}}(y)$$

$$= \alpha s^n \sum_{y \in L^*} e^{-\pi \|sy\|^2} \leq \alpha s^n \sum_{y \in L^*} e^{-\pi \|y\|^2}$$

$$= \alpha s^n \sum_{y \in L^*} \rho(y) \underset{(1)}{=} s^n \rho(L)$$

Divide through by α, get Lemma.

Lemma 3 For any $x \in \mathbb{R}^n$, any lattice L,

$$\rho\left((L+x) \setminus B(0, \sqrt{n}) \right) \leq \frac{1}{2^n} \rho(L).$$

Pf: Consider $\rho_2(L+x)$ (for any L, x)

$$2^n \rho(L) \overset{\text{Lemma 2, } s=2}{\geq} \rho_2(L+x) \geq \rho_2\left((L+x) \setminus B(0, \sqrt{n}) \right)$$

$$= \sum_{y \in L+x,\ \|y\| \geq \sqrt{n}} e^{-\pi \|\frac{y}{2}\|^2} = \sum_{y \in L+x,\ \|y\| \geq \sqrt{n}} e^{\frac{3}{4}\pi \|y\|^2} \cdot e^{-\pi \|y\|^2}$$

$$\geq e^{\frac{3}{4}\pi n} \sum_{\substack{\|y\| \geq \sqrt{n} \\ y \in L+x}} e^{-\pi \|y\|^2} = \left(e^{\frac{3}{4}\pi}\right)^n \rho\left((L+x) \setminus B(0,\sqrt{n})\right)$$

$$\geq 4^n \rho\left((L+x) \setminus B(0,\sqrt{n})\right)$$

Re-arranging gives Lemma 3.

<u>Cor</u>: If L satisfies $\lambda_1(L) > \sqrt{n}$,

$$\rho(L \setminus \{0\}) \leq \frac{1}{2^n} \cdot \frac{1}{1-2^{-n}} \leq \frac{1}{2^{n-1}} .$$

<u>Pf</u> $\rho(L \setminus \{0\}) \underset{\lambda_1(L) > \sqrt{n}}{=} \rho(L \setminus B(0,\sqrt{n}))$

$$\underset{\text{Lemma 3}}{\leq} 2^{-n} \rho(L) \underset{\rho(0)=1}{=} 2^{-n}\left(1 + \rho(L \setminus \{0\})\right)$$

Re-arranging gives the corollary.

Lemma 4 If $\lambda_1(L) > \sqrt{n}$, then for all $x \in \mathbb{R}^n$

$$\left| \frac{\rho(L^* + x)}{\mathrm{covol}(L)} - 1 \right| \leq \frac{1}{2^{n-1}}.$$

Pf: By (2)

$$\frac{\rho(L^* + x)}{\mathrm{covol}(L)} = \sum_{y \in L} \rho(y) e(y \cdot x)$$

$$= 1 + \sum_{y \in L \setminus \{0\}} \rho(y) e(y \cdot x)$$

Second summand satisfies

$$\left| \sum_{y \in L \setminus \{0\}} \rho(y) e(y \cdot x) \right| \underset{\rho(y) > 0}{\leq} \sum_{y \in L \setminus \{0\}} \rho(y) \underbrace{|e(y \cdot x)|}_{\leq 1}$$

$$\leq \rho(L \setminus \{0\}) \underset{\text{cor.}}{\leq} \frac{1}{2^{n-1}}$$

This gives Lemma 4.

Pf of thm Assume by contradiction that for some L, $\lambda_1(L) \cdot \text{covrad}(L^*) > n$.

Rescaling L by a factor c rescales L^* by a factor $\frac{1}{c}$. So by rescaling, we can assume $\lambda_1(L) = \text{covrad}(L^*) > \sqrt{n}$.

By Lemma 4, the function $x \longmapsto \rho(L^* + x)$ is very close to a constant for all x, up to error $\frac{1}{2^{n-1}}$. That is, for all $x, y \in \mathbb{R}^n$, $(\alpha = \text{covol}(L))$

$$\frac{\rho(L^* + y)}{\rho(L^* + x)} \leq \frac{\alpha\left(1 + \frac{1}{2^{n-1}}\right)}{\alpha\left(1 - \frac{1}{2^{n-1}}\right)} \leq 2.$$

Since $\text{covrad}(L^*) > \sqrt{n}$, there is $x \in \mathbb{R}^n$, s.t. $\mathrm{dist}(x, L^*) > \sqrt{n}$, in other words,

any $y \in L^* - x$ satisfies $\|y\| > \sqrt{n}$.

By lemma 3,

$$\rho(L^* - x) = \rho\big((L^* - x) \setminus B(0, \sqrt{n})\big) < 2^{-n} \rho(L^*)$$

$$\frac{\rho(L^*)}{\rho(L^* - x)} > 2^n \qquad \text{a contradiction.}$$