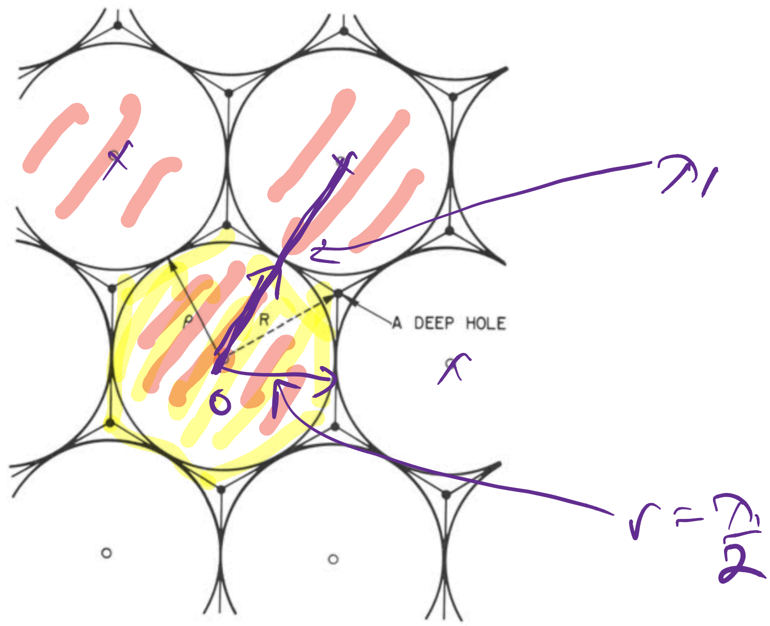
Voronoi cell

ball in packing



(d)

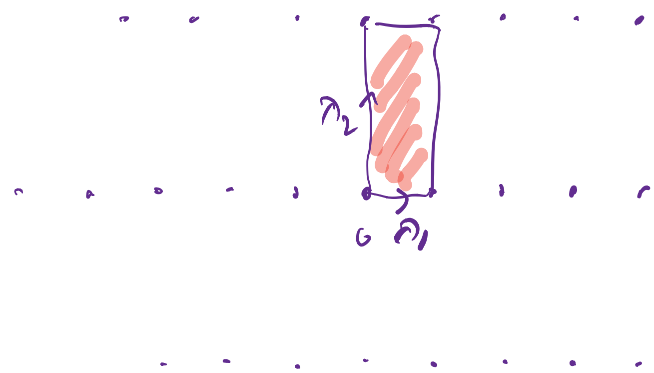
Figure 1.3 (cont.)

$$= \frac{\text{volume of one sphere}}{\text{volume of fundamental region}}$$

Picture: Conway and Sloane, Sphere packings, lattices and groups '88.

Lattices – Lecture 3

Reminder: Minkowski successive minima.



$$\lambda_i(L) = \inf\{r > 0 : \dim \operatorname{span}_{\mathbb{R}}(L \cap B(0,r)) \geq i\}.$$

$\exists\, v_1, \dots, v_n \in L \quad \|v_i\| = \lambda_i(L)$

For a given norm we will sometimes write

$\lambda_i^{\|\cdot\|}(L)$ or $\lambda_i^{K}(L)$ where

$K = B(0,1)$ ← for $\|\cdot\|$.

If we omit $\|\cdot\|$ or K, we usually mean ℓ_2 norm.

Minkowski's second theorem.

For any $L \subset \mathbb{R}^n$ a lattice, and any norm

$$\frac{2^n}{n!}\frac{1}{\mathrm{vol}(B(0,1))} \leq \frac{\lambda_1^{\|\cdot\|}(L)\cdots\lambda_n^{\|\cdot\|}}{\mathrm{covol}(L)} \leq \frac{2^n}{\mathrm{Vol}(B(0,1))}$$

We will prove. For the ℓ_2-norm, have

$$1 \leq \frac{\lambda_1(L)\cdots\lambda_n(L)}{\mathrm{covol}(L)} \leq \frac{2^n}{\mathrm{Vol}(B(0,1))}$$

Remark: For ℓ_2 norm, $\mathrm{Vol}(B(0,1)) \sim \frac{1}{\sqrt{n\pi}}\left(\frac{2\pi e}{n}\right)^{\frac{n}{2}} \approx \frac{1}{\lceil\frac{n}{2}\rceil!}$

sometimes denoted by V_n.

<u>Pf</u>

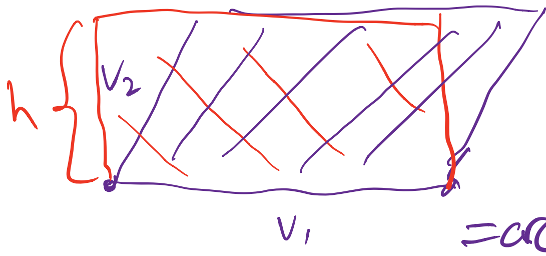
For the LHS we will need the Hadamard inequality: If $A = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$

($v_1, \dots, v_n$ are columns of A) then

$$|\det(A)| \leq \|v_1\| \cdots \|v_n\|.$$

We will prove this as part of a general formalism later. It's a good exercise.

Picture, $n=2$



$|\det(A)| =$

$=$ area of parallelogram

$=$ area of rectangle

$= \|v_1\| \cdot h \leq \|v_1\| \cdot \|v_2\|$

Assuming Hadamard inequality, let $v_1, \ldots, v_n \in L$ linearly independent s.t. $\lambda_i = \|v_i\|$.

Let $L_0 = \mathrm{span}_{\mathbb{Z}}(v_1, \ldots, v_n) \subset L$ $\quad A = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$

so $L_0 = A\mathbb{Z}^n$

$\mathrm{covol}(L) \leq \mathrm{covol}(L_0) = |\det(A)| \leq \|v_1\| \cdots \|v_n\| = \lambda_1 \cdots \lambda_n$

$\mathrm{covol}(L_0) = \mathrm{covol}(L) \cdot [L : L_0]$

Hadamard

Dividing by $\mathrm{covol}(L)$, get LHS.

<u>pf of RHS</u>: Let $v_1, \ldots, v_n$ s.t. $\|v_i\| = \lambda_i$, $v_i \in L$, $v_1, \ldots, v_n$ lin. ind. Note $v_1, \ldots, v_n$ not necessarily a basis for L. By theorem in lecture 1, exists $u_1, \ldots, u_n \in L$ a basis of L, so that

$v_1 = m_{11} u_1$ where $m_{ij} \in \mathbb{Z}$

$v_2 = m_{21} u_1 + m_{22} u_2$ $m_{ii} > 0.$

$\vdots$

$v_n = m_{n1} u_1 + \cdots + m_{nn} u_n$

Claim: If $w \in L$, $\|w\| < \lambda_i$ then $w \in \operatorname{span}_{\mathbb{Z}}(u_1, \dots, u_{i-1})$

Pf of claim By def. of λ_i, $w \in \operatorname{span}_{\mathbb{R}}(v_1, \dots, v_{i-1})$.

Each $v_i \in \operatorname{span}_{\mathbb{R}}(u_1, \dots, u_i)$

Thus $w \in \operatorname{span}_{\mathbb{R}}(u_1, \dots, u_{i-1})$. But u_i's are a basis, so $w \in \operatorname{span}_{\mathbb{Z}}(u_1, \dots, u_{i-1})$.

Apply Gram-Schmidt orthogonalization procedure to $u_1, \dots, u_n$. Get $w_1, \dots, w_n$, orthonormal, such that for each i, $\operatorname{span}(u_1, \dots, u_i) = \operatorname{span}(w_1, \dots, w_i)$.

Let t_{ij} be the coefficients s.t.

$u_1 = t_{11} w_1$

$u_2 = t_{21} w_1 + t_{22} w_2$

$\vdots$

$u_n = t_{n1} w_1 + \cdots + t_{nn} w_n$

For any integer scalars $a_1, \dots, a_n$

$$(*) \quad \left\|\sum_{j=1}^{n} a_j u_j\right\|^2 = \left\|\sum_i \sum_{j\geq i} a_j t_{ji} w_i\right\|^2$$

$$= \sum_i \Big(\sum_{j\geq i} a_j t_{ji}\Big)^2$$

Pythagorean thm, this is ℓ_2 norm.

Now if $(a_1, \dots, a_n) \in \mathbb{Z}^n \setminus \{0\}$, we claim

$$(**) \quad \sum_{i=1}^{n} \frac{1}{\lambda_i^2} \Big(\sum_{j\geq i} a_j t_{ji}\Big)^2 \geq 1$$

<u>Pf of claim</u>: Let j_0 be the last index for which $a_i \neq 0$, i.e. $a_{j_0} \neq 0$, $a_{j_0+1} = \cdots = a_n = 0$

Then $\sum_{j=1}^{n} a_j u_j$ is lin. ind. of $u_1, \dots, u_{j_0-1}$ because $a_{j_0} \neq 0$. Therefore

$$\left\|\sum_{j=1}^{j_0} a_j u_j\right\|^2 = \left\|\sum_{j=1}^{n} a_j u_j\right\|^2 \geq \lambda_{j_0}^2 .$$

In both of $(*)$ and $(**)$ we can replace n by j_0 because $a_{j_0+1} = \cdots = a_n = 0$.

So LHS of $(**)$ is

$$\sum_{i \le j_0} \frac{1}{\lambda_i^2}\Big(\sum_{j \ge i} a_j t_{ji}\Big)^2 \ge \sum_{i \le j_0} \frac{1}{\lambda_{j_0}^2}\Big(\sum_{j \ge i} a_j t_{ji}\Big)^2$$

$$= \frac{1}{\lambda_{j_0}^2} \Big\|\sum_j a_j u_j\Big\|^2 \ge 1, \text{ proving } (**).$$

$(*)$

Let $L' = \mathrm{span}_{\mathbb{Z}}(v_1', \dots, v_n')$, where

$$v_i' = \frac{t_{j1}}{\lambda_1} w_1 + \dots + \frac{t_{jj}}{\lambda_j} w_j$$

Then $(**)$ shows that any nonzero vector of L' has length ≥ 1.

So Voronoi cell of L' contains $B(0, \frac{1}{2})$.

L' is obtained from L by a triangular matrix, of det $\frac{1}{\lambda_1} \cdots \frac{1}{\lambda_n}$.

So $\mathrm{covol}(L') = \dfrac{1}{\lambda_1 \cdots \lambda_n} \mathrm{covol}(L)$

$$\mathrm{covol}(L) = \lambda_1 \cdots \lambda_n \,\mathrm{covol}(L') =$$

$$= \lambda_1 \cdots \lambda_n \,\mathrm{Vol}(\text{Voronoi cell for } L')$$

$$\geq \lambda_1 \cdots \lambda_n \,\mathrm{Vol}(B(0, \tfrac{1}{2})) = \frac{\lambda_1 \cdots \lambda_n}{2^n} \mathrm{Vol}(B(0,1)).$$

Re-arranging: $\dfrac{\lambda_1 \cdots \lambda_n}{\mathrm{covol}(L)} \leq \dfrac{2^n}{\mathrm{Vol}(B(0,1))}$.

Qu What are the optimal bounds for $\lambda_1 \cdots \lambda_n$ for a fixed n and a fixed norm?

Conj (Davenport '46) Let $L \subset \mathbb{R}^n$ a lattice, $\|\cdot\|$ a norm, let $\delta = \delta_{\|\cdot\|}$ be the optimal lattice packing density for $\|\cdot\|$. Then

$$\frac{\lambda_1^{\|\cdot\|}(L) \cdots \lambda_n^{\|\cdot\|}(L)}{\mathrm{covol}(L)} \leq \delta_{\|\cdot\|} \frac{2^n}{\mathrm{Vol}(B(0,1))}$$

$$\delta_{\|\cdot\|}(L) = \sup \left\{ \frac{\mathrm{Vol}(B(0,r))}{\mathrm{covol}(L)} : B(\ell, r) \text{ are disjoint for distinct } \ell \right\}$$

w.r.t. $\|\cdot\|$

$$= \frac{\mathrm{Vol}(B(0, \frac{\lambda(L)}{2}))}{\mathrm{covol}(L)} = \left(\frac{\lambda(L)}{2}\right)^n \frac{\mathrm{Vol}(B(0,1))}{\mathrm{covol}(L)}$$

This is the packing density of L w.r.t. $\|\cdot\|$.

$$\delta_{\|\cdot\|} = \sup\left\{ \delta_{\|\cdot\|}(L) : \begin{array}{c} L \subset \mathbb{R}^n \\ \text{a lattice} \end{array} \right\}$$

$$= \max\left\{ \delta_{\|\cdot\|}(L) : L \subset \mathbb{R}^n \text{ a lattice} \right\} \leq 1$$

we'll prove later.

Understanding $\delta_{\|\cdot\|}$ is a wide open question. Included in Hilbert's list of problems 1900.

Wide open for ℓ_2 balls.

Denote by δ_n the optimal lattice packing density for ℓ_2-norm in $\mathbb{R}^n$.

$$\frac{1}{2^n} \leq \delta_n$$

(Minkowski – Hlawka-Siegel, we will prove).

$$\frac{65963 \cdot n}{2^n} \leq \delta_n$$

all large n

Venkatesh 2018, following work of many people.

$$\delta_n \leq 0.67^n$$

all large n, Kabatyanski & Levenshtein '78

<u>Reminder:</u> Minkowski's first thm

If L is a lattice, K is centrally symmetric and convex, $\mathrm{Vol}(K) > 2^n \mathrm{covol}(L)$ then $L \cap K \neq \{0\}$.

Relationship between 1st and 2nd theorems?

<u>Claim:</u> First theorem is essentially equivalent to

$$\frac{\lambda_1^n}{\mathrm{covol}(L)} \leq \frac{2^n}{\mathrm{Vol}(B(0,1))} \qquad (*)$$

Pf that Minkowski 1 $\Rightarrow$ $(*)$.

Given L, define r_0 be the formula

$$2^n \operatorname{covol}(r_0 L) = \operatorname{Vol}(K).$$

$$\Updownarrow$$

$$r_0^n = \frac{\operatorname{Vol}(K)}{2^n \operatorname{covol}(L)}.$$

If $r < r_0$, can apply Minkowski's 1^{st} thm, to obtain $rL \cap K \neq \{0\}$.

$\Longleftrightarrow$ $\lambda_1^K(rL) < 1$

Letting $r \to r_0$ $\quad \underbrace{\lambda_1^K(r_0 L)}_{= r_0 \lambda_1^K(L)} \leq 1$

$$(\lambda_1^K(L))^n \leq \frac{1}{r_0^n} = \frac{2^n \operatorname{covol}(L)}{\operatorname{Vol}(K)}$$

Applications of Minkowski 1^{st} thm

(I) Application in Diophantine approximation.

Thm (Dirichlet) $\forall x \in \mathbb{R}^d$ $\forall Q > 1$ $\exists p \in \mathbb{Z}^d$ $q \in \mathbb{N}$ s.t.

$$\left\| x - \frac{1}{q}\cdot p \right\|_\infty \le \frac{1}{q Q^{1/d}}$$

and $\quad 1 \le q \le Q$.

If $d=1$ $\quad \left| x - \frac{p}{q} \right| \le \frac{1}{qQ} \le \frac{1}{q^2}$

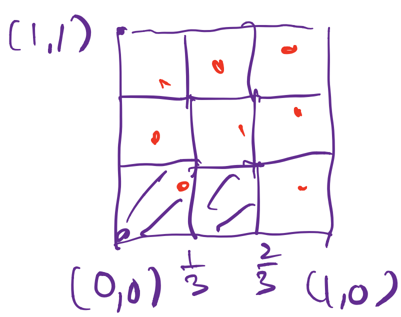
First proof (works for $Q = T^d$, $T \in \mathbb{N}, T > 1$)

Subdivide $[0,1)^d$ into $Q = T^d$ subcubes of sidelength $\frac{1}{T}$.

$$\prod_{i=1}^{d} \left[a_i, a_i + \tfrac{1}{T}\right) \qquad \text{where } a_i \in \left\{0, \tfrac{1}{T}, \ldots, 1 - \tfrac{1}{T}\right\}.$$

$i = 1, \ldots, d$.

Picture $d=2$, $T=3$



Look at $\quad 0, \{x\}, \{2x\}, \ldots, \{Qx\}$

where $\left\{\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}\right\} = \begin{pmatrix} \{x_1\} \\ \vdots \\ \{x_d\} \end{pmatrix}$ $\quad \{x\} = x - \lfloor x \rfloor$.

These are $Q+1$ points, and the partition of $[0,1]^d$ contains Q cells. So there is

$0 \le i < j \le Q$ s.t. $\{ix\}, \{jx\}$ belong to same cell.

$$\{ix\} = ix - p_1 \qquad p_1 \in \mathbb{Z}^d$$
$$\{jx\} = jx - p_2 \qquad p_2 \in \mathbb{Z}^d$$

$$\frac{1}{T} \ge \|\{jx\} - \{ix\}\|_\infty = \|jx - p_2 - (ix - p_1)\|_\infty =$$
$$= \|(j-i)x - (p_2 - p_1)\|_\infty$$

set $q = j - i \in \{1, \dots, Q\}$
$p = p_2 - p_1 \in \mathbb{Z}^d$

$$\|qx - p\|_\infty \le \frac{1}{T} \qquad \|x - \tfrac{1}{q}p\| \le \frac{1}{qQ^{\frac{1}{d}}}.$$

2^{nd} proof (using Minkowski) Want:

$$\begin{cases} \|x - \frac{1}{q}\cdot p\| \le \frac{1}{q Q^{\frac{1}{d}}} \\ q \in \mathbb{N} \quad q \le Q \end{cases} \iff \|qx - p\| \le Q^{-\frac{1}{d}}$$

$$(p \in \mathbb{Z}^d, q \in \mathbb{N})$$

$$\iff \begin{cases} \|qx - p\|_\infty \le Q^{-\frac{1}{d}} \\ q \in \mathbb{Z},\ q \neq 0 \quad |q| \le Q \end{cases}$$

$\iff \exists\, (p, q) \in \mathbb{Z}^{d+1}$, $q \neq 0$ in

$$K = \left\{ \begin{pmatrix} y_1 \\ \vdots \\ y_d \\ y_{d+1} \end{pmatrix} \in \mathbb{R}^{d+1} : \begin{array}{l} \underbrace{|y_{d+1}|}_{y_{d+1} \text{ in an interval of size } 2Q} \le Q \\ \underbrace{|y_{d+1} x_i - y_i|}_{y_i \text{ satisfying ineq. are in } [x_i y_{d+1} - Q^{-\frac{1}{d}},\ x_i y_{d+1} + Q^{-\frac{1}{d}}]} \le Q^{-\frac{1}{d}} \quad i = 1, \dots, d \end{array} \right\}$$

Take $L = \mathbb{Z}^{d+1}$ covol $(L) = 1$

$\mathrm{Vol}(K) = 2^{d+1}$ — repeated integration

$$\mathrm{Vol}(K) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \mathbb{1}_K(y_1, \dots, y_{d+1})\, dy_{d+1}\, dy_1 \cdots dy_d\,.$$

$$= 2Q\left(2Q^{-\frac{1}{d}}\right)^d = 2^{d+1}.$$

Minkowski implies there is $\binom{p}{q} \in \mathbb{Z}^{d+1} \setminus \{0\}$

in K.

If this solution has $q=0$ then p_i's satisfy $|p_i| \le Q^{-\frac{1}{d}} < 1$

so $p_i = 0$.

If there is a nonzero solution, (p,q), then necessarily $q \neq 0$.

<u>Extension</u> Minkowski's linear forms thm.

Let $m, n \in \mathbb{N}$ $\quad (a_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$

$L_j : \mathbb{R}^m \to \mathbb{R}$ linear functionals given by

$$L_j \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \sum_{i=1}^{m} a_{ij} x_i .$$

(so have n linear functionals on $\mathbb{R}^m$).

Then for each $Q > 1$ $\exists\, q \in \mathbb{Z}^m \setminus \{0\}$, and $p \in \mathbb{Z}^n$ s.t. for each j,

$$(*) \quad |L_j(q) - p_j| \leq Q^{-\frac{m}{n}}, \quad \text{and } \|q\|_\infty \leq Q.$$

<u>Remark:</u> if $m=1$, $n=d$ get Dirichlet's thm.

<u>Pf</u> Use Minkowski's thm as above, with

$$L = \mathbb{Z}^{m+n}, \quad K = \left\{ \begin{pmatrix} y_1 \\ \vdots \\ y_n \\ z_1 \\ \vdots \\ z_m \end{pmatrix} \in \mathbb{R}^{n+m} : \begin{array}{l} \left\| \begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix} \right\|_\infty \leq Q \\ \left| L_j\begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix} - y_j \right| \leq Q^{-\frac{n}{m}} \\ j = 1, \dots, m \end{array} \right\}$$

Get that $\exists \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{Z}^{n+m} \setminus \{0\}$ s.t. $(*)$ holds. If $q = 0$ then $|p_j| \leq Q^{-\frac{n}{m}} < 1$

so $p_j = 0$. So must have $q \neq 0$.

Minkowski's discriminant bound (special case).

Let $P \in \mathbb{Z}[X]$, i.e. $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$

$a_i \in \mathbb{Z}$. Assume P is <u>monic</u>, i.e. $a_n = 1$.

P is called <u>irreducible</u> (over $\mathbb{Q}$) if there are no $P_1, P_2 \in \mathbb{Q}[X]$ s.t. $\deg P_i < \deg P$ $i=1,2$.

s.t. $P = P_1 \cdot P_2$.

If P is irreducible and α is a root of P then $\deg P$ is $\deg(\alpha)$, i.e. if $Q \in \mathbb{Z}[X]$, $Q(\alpha) = 0 \Rightarrow \deg P \leq \deg Q$.

[Pf: If $P_1, P_2 \in \mathbb{Q}[X]$, $\deg P_2 > \deg P_1$ then $\exists Q, R \in \mathbb{Q}[X]$ s.t.

$$P_2 = QP_1 + R, \quad \deg R < \deg P_1].$$

Also, roots of P are distinct.

If $P(\alpha) = 0$, $(x-\alpha)^2 \mid P$ then $P'(\alpha) = 0$, contradicting previous fact.

If $\xi_1, \dots, \xi_n$ are the roots of P, then
$$\Delta = \prod_{1 \leq i < j \leq n} (\xi_i - \xi_j)^2$$
is the discriminant of P. (does not depend on how we order the roots).

So Δ is symmetric in the roots of P (does not change if we permute the roots).

Fact: If $Q \in \mathbb{Z}[x_1, \ldots, x_n]$ is a polynomial in n vars which is symmetric in the roots of P (i.e. $Q(\xi_1, \ldots, \xi_n) = Q(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(n)})$ for any permutation $\sigma \in S_n$) then

$$Q(\xi_1, \ldots, \xi_n) \in \mathbb{Z}.$$

Pf Idea: ξ_i are algebraic integers (because P is monic). Algebraic integers form a ring $\Rightarrow Q(\xi_1, \ldots, \xi_n)$ also an alg. integers. The symmetry property ensures that $Q(\xi_1, \ldots, \xi_n)$ is inv. under the Galois group of $\mathbb{Q}(\xi_1, \ldots, \xi_n)/\mathbb{Q}$.

$\Rightarrow Q(\xi_1, \ldots, \xi_n) \in \mathbb{Q}$

$\Rightarrow$ any alg. int. in $\mathbb{Q}$ is in $\mathbb{Z}$.

<u>Thm</u> Suppose all ξ_i are real, then

$$\Delta \geq \left(\frac{n^n}{n!}\right)^2.$$

<u>Pf</u>: Define $y_j \in (\mathbb{R}^n)^*$ linear functional on $\mathbb{R}^n$

$$y_j\begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix} = \sum_{i=1}^{n} \xi_j^{i-1} x_i \quad . \quad (P(\xi_j) = 0 \quad j=1,\dots,n)$$

We will be interested in the values of $y_j(x)$, for $x \in \mathbb{Z}^n$.

If $y_j(x) = 0$ for some $x \in \mathbb{Z}^n$, then we get a polynomial in $\mathbb{Z}[X]$ for which ξ_j is a root, (x_i) are coeff., and $\deg < n$.

So $x = 0$.

So for all $x \in \mathbb{Z}^n \setminus \{0\}$,

$|y_1(x)\cdots y_n(x)| > 0.$

But $Q(\xi_1,\dots,\xi_n) = y_1(x)\cdots y_n(x)$
(x fixed, think of this as a fn of ξ_j).

Q is symmetric in $\xi_1,\dots,\xi_n$.

So $|y_1(x)\cdots y_n(x)| \geq 1$, $\forall x \in \mathbb{Z}^n \setminus \{0\}$.

Define $\left\|\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}\right\| = \frac{1}{n}\sum_{j=1}^{n} |z_j|$ (ℓ_1-norm divided by n)

$$A = \begin{pmatrix} 1 & \xi_1 & \cdots & \xi_1^{n-1} \\ \vdots & & & \\ 1 & \xi_2 & & \xi_n^{n-1} \end{pmatrix}$$

$$L = A\mathbb{Z}^n = \left\{ \begin{pmatrix} y_1(x) \\ \vdots \\ y_n(x) \end{pmatrix} : x \in \mathbb{Z}^n \right\}$$

$\det(A)^2 = \Delta$ (Van der Monde)

By inequality of means:

$$\left\| \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \right\| = \frac{1}{n}\sum_{i=1}^{n} |z_i| \geq \left(\prod_{i=1}^{n} |z_i|\right)^{\frac{1}{n}}$$

Use with with $y \in L \setminus \{0\}$ $y = Ax$, $x \in \mathbb{Z}^n \setminus \{0\}$.

$$\left\| \begin{pmatrix} y_1(x) \\ \vdots \\ y_n(x) \end{pmatrix} \right\| \geq \left|\prod_{i=1}^{n} y_i(x)\right|^{\frac{1}{n}} \geq 1$$

So $B(0,1) \cap L = \{0\}$.

# of sectors

$$V = Vol(B(0,1)) = \frac{2^n \cdot n^n}{n!}$$

Recall: $Vol(conv(0, e_1, \dots, e_n)) = \frac{1}{n!}$

$$B(0,1) = conv(\pm ne_1, \pm ne_2, \dots, \pm ne_n)$$

By Minkowski, with $K = B(0,1)$

$$2^n covol(L) \geq V$$

$$2^n \Delta^{1/2} \geq \frac{2^n n^n}{n!} \iff \Delta \geq \left(\frac{n^n}{n!}\right)^2$$

We have $\lambda_1, \dots, \lambda_n$ which we think of as measuring the geometry of a lattice.

We will introduce other parameters α_i, defined as follows.

Let $L_0 \subset L$ be a subgroup (w.r.t. vector addition). As we saw, $L_0 = \mathrm{span}_{\mathbb{Z}}(u_1, \dots, u_r)$ $u_1, \dots, u_r \in L$ lin. ind.

L_0 is <u>primitive</u> if there exist $v_{r+1}, \dots, v_n \in L$ s.t. $\mathrm{span}_{\mathbb{Z}}(u_1, \dots, u_r, v_{r+1}, \dots, v_n) = L$.

(if this happens we also $u_1, \dots, u_r$ is primitive).

We saw that L_0 is primitive $\iff$ $L \cap \mathrm{span}_{\mathbb{R}}(L_0) = L_0$. (Lecture 1).

r will be called the rank of L_0, notation $r = \mathrm{rank}(L_0)$.

L_0 is a lattice in $\mathrm{span}_{\mathbb{R}}(L_0) \cong \mathbb{R}^r$

We can normalize Lebesgue measure on $\mathrm{span}_{\mathbb{R}}(L_0)$ by taking Lebesgue measure m on $\mathbb{R}$, normalized so that $m\left(\left\{\sum a_i w_i : a_i \in [0,1)\right\}\right) = 1$,

where $w_1, \dots, w_r$ is an orthonormal basis of $\mathrm{span}_{\mathbb{R}}(L_0)$.

We can define $\mathrm{covol}(L_0)$ to be $m(\Omega)$ where Ω is a fundamental domain for L_0 in $\mathrm{span}_{\mathbb{R}}(L_0)$.

Define $\alpha_i(L) = \inf\left\{\mathrm{covol}(L_0) : \begin{matrix} L_0 \subset L \\ \mathrm{rank}(L_0) = i \end{matrix}\right\}$

$= \inf\left\{\mathrm{covol}(L_0) : \begin{matrix} L_0 \subset L \\ \mathrm{rank}(L_0) = i \end{matrix} \ L_0 \text{ primitive}\right\}$