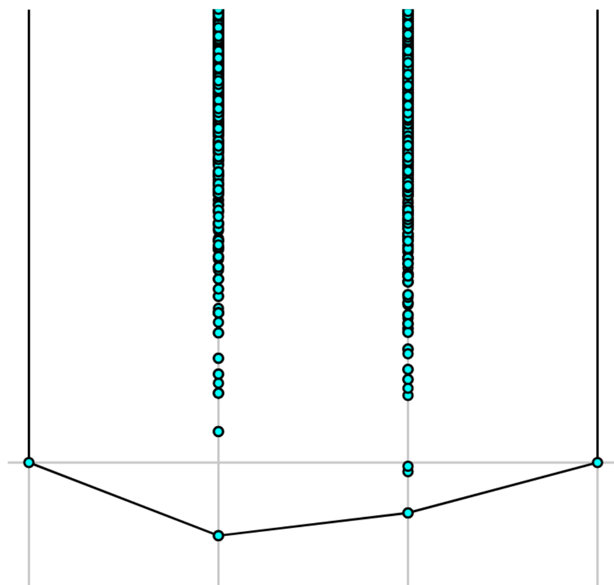
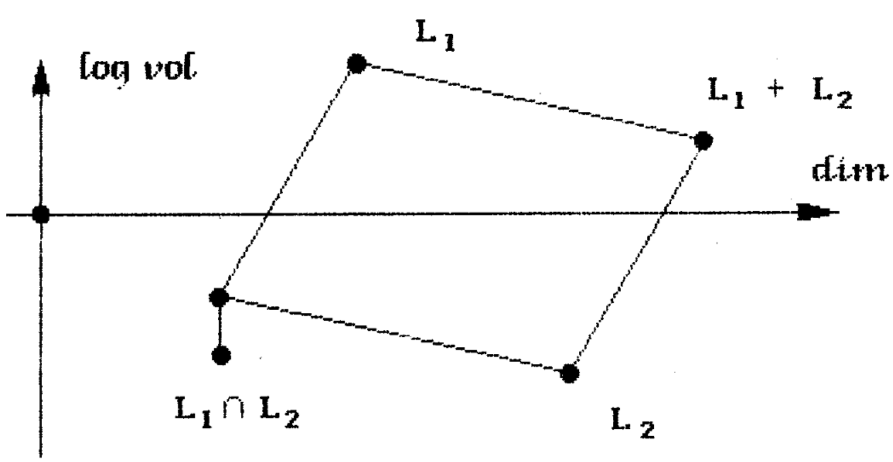
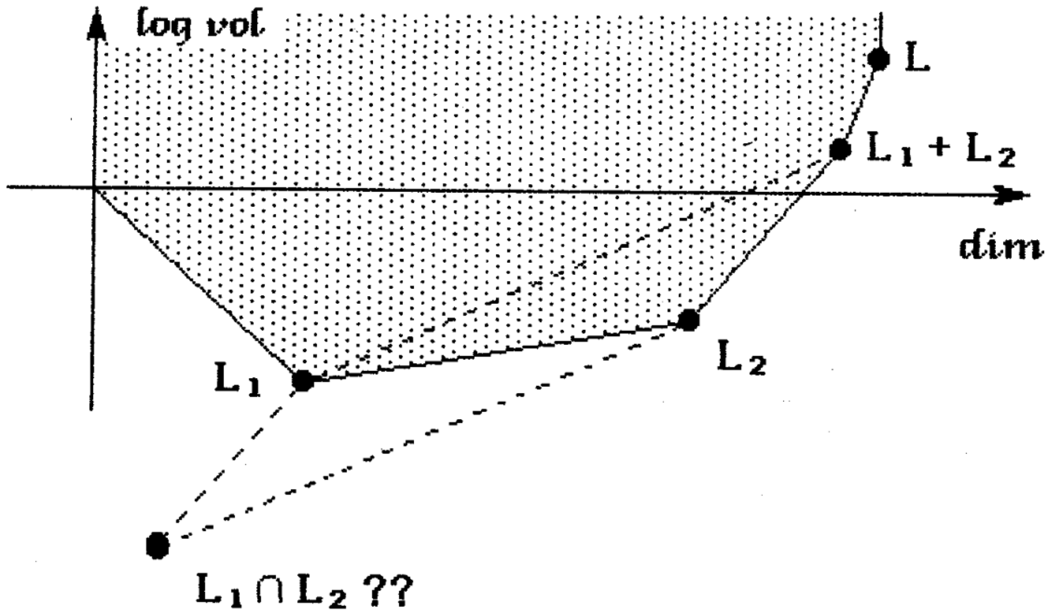


Picture: Casselman, Stability of lattices and the partition of arithmetic quotients, '04



Picture from D. Grayson, Reduction theory via semistability, '84.



Lattices lecture 4

Recall $L \subset \mathbb{R}^n$ a lattice

$L_0 \subset L$ an additive subgroup

$L_0 = \mathrm{span}_{\mathbb{Z}}(u_1, \ldots, u_r)$ $u_1, \ldots, u_r$ linearly independent

r = rank of L_0.

L_0 is primitive if $\exists\, v_{r+1}, \ldots, v_n$ s.t.

$\mathrm{span}_{\mathbb{Z}}(u_1, \ldots, u_r, v_{r+1}, \ldots, v_n) = L$.

$\Longleftrightarrow$ $L \cap \mathrm{span}_{\mathbb{R}}(L_0) = L_0$.
lecture 1

$$\lambda_i(L) = \inf\left\{ \operatorname{covol}(L_0) : \begin{array}{l} L_0 \subset L \text{ additive subgroup} \\ \operatorname{rank}(L_0) = i \end{array} \right\}$$

$$= \inf\left\{ \operatorname{covol}(L_0) : \begin{array}{l} L_0 \subset L \quad L_0 \text{ primitive} \\ \operatorname{rank}(L_0) = i \end{array} \right\}$$

we'll see today

$$= \min \{ \qquad \}$$

we'll see later

Making sense of $\operatorname{covol}(L_0)$.

L_0 is a lattice in $\operatorname{span}_{\mathbb{R}}(L_0) \cong \mathbb{R}^r$

$r = \operatorname{rank}(L_0)$.

$\operatorname{covol}(L_0) = \operatorname{Vol}$ (fund. domain for L_0).

r-dim.

<u>Recall</u>: Lebesgue measure on $\mathbb{R}^k$ (which we denote by Vol)

is the unique Borel measure on $\mathbb{R}^k$, which

is translation invariant (i.e. $\mu(A) = \mu(A+x)$

for any $A \subset \mathbb{R}^k$ Borel

any $x \in \mathbb{R}^k$)

and normalized (i.e. $\mu([0,1)^k)=1$).

$$\begin{tikzcd} & V=\mathrm{span}_{\mathbb{R}}(L_0) & \\ \mathbb{R}^r \arrow[ur, "\varphi"] \arrow[rr, "\psi^{-1}\circ\varphi"'] & & \mathbb{R}^n \arrow[ul, "\psi"'] \end{tikzcd}$$

lin. iso.

If $T:\mathbb{R}^k \to \mathbb{R}^k$ is a linear transformation, then for all $A \subset \mathbb{R}^k$ Borel,

$$\mathrm{Vol}(T(A)) = |\det(T)| \cdot \mathrm{Vol}(A)$$

Jacobian

<u>Solution</u> Define the volume on V to be $\mathrm{Vol}_V(A) = c\,\mathrm{Vol}_{\mathbb{R}^r}(\varphi^{-1}(A))$

where c is chosen so that if $w_1,\dots,w_r \in V$ orthonormal in V then

$$\mathrm{Vol}_V(\{\sum a_i w_i : a_i \in [0,1)\}) = 1.$$

Use Vol_V to define $\mathrm{covol}(L_0)$ as

Vol_V (fund. domain for L_0 in V)

$= \mathrm{Vol}_V(\{ \sum a_i u_i : a_i \in [0,1] \})$

where $L_0 = \mathrm{span}_{\mathbb{Z}}(u_1, \dots, u_r)$.

Note: If L_0 is not primitive,

can define $L_1 = L \cap \mathrm{span}_{\mathbb{R}}(L_0)$.

$\mathrm{covol}(L_0) = \mathrm{covol}(L_1) \cdot [L_1 : L_0]$.

$\mathrm{covol}(L_1) \le \mathrm{covol}(L_0)$.

This explains why $\inf \{ \quad \} = \inf \{ \text{primitive} \}$

in definition of $\alpha_i(L)$.

<u>Convention</u> $\mathrm{covol}(\{0\}) = 1$

$\alpha_0(L) = 1$ for any lattice L.

<u>Prop.</u> For each n $\exists C > 1$ $\forall L$ $\forall i$

$$\frac{1}{C} \lambda_1(L) \cdots \lambda_i(L) \le \alpha_i(L) \le C \lambda_1(L) \cdots \lambda_i(L)$$

<u>Ex</u> $n=3$ $L = \frac{1}{2}\mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus 2\mathbb{Z}e_3$

In this case $\alpha_2 = \mathrm{covol}(\frac{1}{2}\mathbb{Z}e_1 \oplus \mathbb{Z}e_2)$

So have equality $\lambda_1\lambda_2 = \alpha_2$

$\lambda_1 = \alpha_1$

$\lambda_1\lambda_2\lambda_3 = \mathrm{covol}(L) = \alpha_3$

Pf (ex, we will prove for $n=2,3$ to give the idea). Write $A \ll B$ if A, B are functions of a lattice L and there exists a const. C, depending on n but indep. of L, s.t. $\forall L$, $A(L) \leq C B(L)$.

$A \asymp B$ if $A \ll B$, $B \ll A$.

In case $n=2$. $\alpha_1 = \lambda_1$ (by def of α_1)

$\alpha_2 = \mathrm{covol}(L)$ (by def of α_2)

$\asymp \lambda_1\lambda_2$

by Minkowski's second theorem.

In case $n=3$. $\alpha_1 = \lambda_1$

as before $\alpha_3 = \mathrm{covol}(L) \asymp \lambda_1\lambda_2\lambda_3$

So the interesting case is $i=2$.

Take v_1, v_2 in L linearly ind. with

$\|v_1\| = \lambda_1$, $\|v_2\| = \lambda_2$. $L_0 = \mathrm{span}_{\mathbb{Z}}(v_1, v_2)$.

$$\alpha_2(L) \leq \mathrm{covol}(L_0) = \mathrm{Vol}_V\left(\left\{\sum_{i=1}^{2} a_i v_i : a_i \in [0,1)\right\}\right)$$

by def of α_2

$$= \|v_1\| \cdot \|v_2\| \cdot |\sin\theta| \leq \|v_1\| \cdot \|v_2\| = \lambda_1 \lambda_2$$

area of a parallelogram

θ angle at base

So $\alpha_2 \leq \lambda_1 \lambda_2$

Now suppose $L_0 = \mathrm{span}_{\mathbb{Z}}(u_1, u_2)$

s.t. $\alpha_2(L) \geq \frac{1}{2}\mathrm{covol}(L_0)$

$$\mathrm{covol}(L_0) \geq c\lambda_1(L_0)\lambda_2(L_0) \geq c\lambda_1(L)\lambda_2(L)$$

case $n=2$

$$\alpha_2(L) \gg \lambda_1 \lambda_2.$$

We will need some algebraic preliminaries

in order to compute $\mathrm{covol}(L_0)$, for $L_0 \subset L$

of rank r.

Grassmannians, Plücker coordinates, exterior products

Let $e_1, \dots, e_n$ be standard basis of $\mathbb{R}^n$.
Denote the standard inner product on $\mathbb{R}^n$ by $\langle \cdot, \cdot \rangle$. For $p \in \{1, \dots, n\}$ and $\sigma = (1 \le i_1 < \dots < i_p \le n)$ define $e_\sigma = e_{i_1} \wedge \dots \wedge e_{i_p}$ (formal expression).

Let $C(n,p)$ be the collection of such expressions
$|C(n,p)| = \binom{n}{p} = \ell$
Let $\mathbb{R}^n_p$ denote the ℓ-dim real vector space spanned by e_σ. By convention $\mathbb{R}^n_0 = \mathbb{R}$.
$\mathbb{R}^n_0 = \text{span}(1)$.
The elements of $\mathbb{R}^n_p$ are called <u>p-vectors</u>.
Note $\mathbb{R}^n_1 = \mathbb{R}^n$.
Denote $G_n = \mathbb{R}^n_0 \oplus \dots \oplus \mathbb{R}^n_n$. The <u>Grassmannian algebra</u>
So $\dim G_n = \sum_{p=0}^{n} \binom{n}{p} = 2^n$.

Define $\langle e_\sigma, e_\tau \rangle = \delta_{\sigma\tau} = \begin{cases} 1 & \sigma = \tau \\ 0 & \sigma \neq \tau \end{cases}$

We will use formal expressions

$$e_{j_1} \wedge \cdots \wedge e_{j_p} \qquad j_1, \ldots, j_p \in \{1, \ldots, n\}$$

and they will be interpreted as follows.

If $j_k = j_\ell$ for some $k \neq \ell$ then

$e_{j_1} \wedge \cdots \wedge e_{j_p} = 0$. $\qquad \boxed{e_5 \wedge e_2 = -e_2 \wedge e_5}$

If $\{j_1, \ldots, j_p\} = \{i_1, \ldots, i_p\}$ where

$1 \leq i_1 < \cdots < i_p \leq n$, and $j_\ell = i_{\pi(\ell)}$ for

some permutation π of p indices,

then $e_{j_1} \wedge \cdots \wedge e_{j_p} = \begin{cases} +e_{i_1} \wedge \cdots \wedge e_{i_p} & \pi \text{ even} \\ -e_{i_1} \wedge \cdots \wedge e_{i_p} & \pi \text{ odd} \end{cases}$

On G_n we have the structure of a real vector space, with an inner product.

We defined a product $\wedge : G_n \times G_n \to G_n$
$(u, v) \longmapsto u \wedge v$

as follows:

$1 \wedge 1 = 1$, $\quad 1 \wedge e_\sigma = e_\sigma \wedge 1 = e_\sigma$

$(e_{i_1} \wedge \cdots \wedge e_{i_p}) \wedge (e_{j_1} \wedge \cdots \wedge e_{j_q}) =$

$= e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_{j_1} \wedge \cdots \wedge e_{j_q}$ (interpreted as before).

Extend $\wedge$ by linearity to all elements of G_n: $\left(\sum_\sigma a_\sigma e_\sigma\right) \wedge \left(\sum_{\sigma'} b_{\sigma'} e_{\sigma'}\right) =$

$$= \sum_{\sigma,\sigma'} a_\sigma b_{\sigma'} e_\sigma \wedge e_{\sigma'}$$

With this structure G_n is a unital associative algebra: V is a unital associative algebra if V is a vector space, equipped with a product $V \times V \to V$, which is associative, multilinear, and contains a unit (neutral element for multi)

$(x+y)\cdot z = x \cdot z + y \cdot z \qquad (ax)\cdot(by) = (ab)\, x \cdot y$

$z \cdot (x+y) = z \cdot x + z \cdot y$

If $x_1, \dots, x_p \in \mathbb{R}^n_1 = \mathbb{R}^n$ then

$x_1 \wedge \cdots \wedge x_p \in \mathbb{R}^n_p$ is called <u>decomposable</u> or

a <u>p-blade</u>.

<u>Prop</u> $x_i = \sum_{j=1}^{n} \xi_{i,j} e_j$ $\quad i = 1, \dots, p$ $\quad \xi_{i,j} \in \mathbb{R}$

$$x_i = \begin{pmatrix} \xi_{i,1} \\ \vdots \\ \xi_{i,n} \end{pmatrix}$$

Then

$$(*) \quad x_1 \wedge \cdots \wedge x_p = \sum_{\sigma \in C(n,p)} \xi_\sigma e_\sigma, \text{ where } \xi_\sigma = \det \left(\xi_{ij} \right)_{\substack{i=1,\dots,p \\ j \in \sigma}}$$

<u>Example</u> $n=4 \quad p=2$

$$\begin{pmatrix} \xi_{1,1} \\ \vdots \\ \xi_{1,4} \end{pmatrix} \wedge \begin{pmatrix} \xi_{2,1} \\ \vdots \\ \xi_{2,4} \end{pmatrix} = \left(\xi_{1,1} \cdot \xi_{2,2} - \xi_{1,2} \xi_{2,1} \right) e_{1,2} +$$

$$\cdots + \left(\xi_{1,2} \cdot \xi_{2,3} - \xi_{1,3} \xi_{2,2} \right) e_{2,3} + \cdots + \left(\xi_{13} \xi_{24} - \xi_{14} \xi_{23} \right) e_{3,4}$$

<u>Pf</u>: Both sides of $(*)$ transform identically if we perform linear operations on the x_i (for each individual i).

So suffices to check $(*)$ when x_i are e_{j_i} for some j_i. Both sides transform identically when permuting the x_i, so can assume $1 \le j_1 \le \cdots \le j_p \le n$.

If $j_i = j_{i+1}$ for some i, the LHS of $(*) = 0$.

The RHS is also zero because each matrix (ξ_{ij}) has two identical rows.

If $1 \le j_1 < \cdots < j_p \le n$, then LHS is e_{σ_0} where $\sigma_0 = (1 \le j_1 < \cdots < j_p \le n)$.

RHS, coefficient of e_{σ_0} is $\det(I_p) = 1$, coefficient of e_σ for $\sigma \ne \sigma_0$ is det of a matrix with a 0 column, hence is 0.

So $\underline{\text{LHS} = \text{RHS}}$.

Cor 1 $x_1 \wedge \cdots \wedge x_p \neq 0 \iff x_1, \ldots, x_p$ are lin. ind.

Pf Immediate from $(*)$. (ex.)

Cor 2 $\forall x \in \mathbb{R}^n$, $x \wedge x = 0$.

Pf Immediate from Cor 1.

Def Say that $v_1, v_2 \in G_n \setminus \{0\}$ are proportional if $\exists c$ s.t. $v_2 = c v_1$.

Prop: Suppose $x_1, \ldots, x_p$ linearly independent and $y_1, \ldots, y_p$ linearly ind.

Then $x_1 \wedge \cdots \wedge x_p$ are $y_1 \wedge \cdots \wedge y_p$ are proportional

$$\Updownarrow$$

$$\operatorname{span}_{\mathbb{R}}(x_1, \ldots, x_p) = \operatorname{span}_{\mathbb{R}}(y_1, \ldots, y_p).$$

Pf $\Uparrow$ Let $V = \operatorname{span}_{\mathbb{R}}(x_1, \ldots, x_p)$

$= \operatorname{span}_{\mathbb{R}}(y_1, \ldots, y_p)$

Each y_i is a lin. comb. of x_j

$$y_i = \sum_{j=1}^{p} a_{ji} x_j$$

$$y_1 \wedge \cdots \wedge y_p = \left(\begin{array}{c}\text{lin. comb} \\ \text{of } x_j\end{array}\right) \wedge \left(\begin{array}{c}\text{lin. comb} \\ x_j\end{array}\right) \wedge \cdots \wedge (\quad)$$

$= \text{multiple of } x_1 \wedge \cdots \wedge x_p$

manipulate using mult. linearity

$\Downarrow$ Since $y_1 \wedge \cdots \wedge y_p \neq 0$

$y_1 \wedge \cdots \wedge y_p = c\, x_1 \wedge \cdots \wedge x_p$, $c \neq 0$

$$y_i \wedge (x_1 \wedge \cdots \wedge x_p) = \frac{1}{c} y_i \wedge (y_1 \wedge \cdots \wedge y_p)$$

$= 0$. Therefore, by Cor. 1

from Cor. 2

$y_i, x_1, \ldots, x_p$ are lin. dependent

but $x_1, \ldots, x_p$ are linearly independent.

i.e. $y_i \in \operatorname{span}(x_j)$.

This is true for all i, and hence

$\operatorname{span}(y_j) \subset \operatorname{span}(x_j)$

By comparing dimensions, $\mathrm{Span}_{\mathbb{R}}(y_i) = \mathrm{span}_{\mathbb{R}}(x_j)$

<u>Rk</u>: Previous prop gives a map

$$\mathrm{Gr}_{n,p}(\mathbb{R}) = \left\{ \begin{array}{c} p \text{ dimensional linear subspaces} \\ \text{of } \mathbb{R}^n \end{array} \right\}$$

$$\downarrow$$

$$\mathbb{P}(\mathbb{R}^n_p) = \left\{ \begin{array}{c} \text{lines through the origin} \\ \text{in } \mathbb{R}^n_p \end{array} \right\}$$

(mapping is $\mathrm{span}_{\mathbb{R}}(x_1, \dots, x_p) \longmapsto \mathbb{R}\cdot(x_1 \wedge \dots \wedge x_p)$)

inj. and well-defined by the Prop.

map is the Plücker embedding.

<u>Lemma</u> (Laplace identity, special case of Cauchy-Binet formula). Let $x_1, \dots, x_p$, $y_1, \dots, y_p$ vectors in $\mathbb{R}^n$. Then

$$\langle x_1\wedge\cdots\wedge x_p, y_1\wedge\cdots\wedge y_p\rangle = \det\left(\langle x_i, y_j\rangle\right)_{\substack{i=1,\dots,p\\ j=1,\dots,p}}.$$

Pf: First check for

$x_1,\dots,x_p = e_{i_1},\dots,e_{i_p}$ $\quad \sigma = (1\le i_1<\cdots<i_p\le n)$

$y_1,\dots,y_p = e_{j_1},\dots,e_{j_p}$ $\quad \tau = (1\le j_1<\cdots<j_p\le n)$.

If $\sigma=\tau$ LHS is 1 and RHS is $\det(I_p)=1$

If $\sigma\neq\tau$ then for some i, $x_i\neq y_j$ for all j.

Therefore RHS has a zero row and is zero

LHS is also zero.

General case: reduce to this special by row operations. (ex.).

Cor: (a) For any $u_1,\dots,u_k, v$, have

$$\|u_1\wedge\cdots\wedge u_k\wedge v\| \le \|u_1\wedge\cdots\wedge u_k\|\cdot\|v\|.$$

(b) Furthermore, if $u_1,\dots,u_k$, $v_1,\dots,v_l$ satisfy

$\langle u_i, v_j \rangle = 0 \ \forall i,j$, then

$$\|u_1 \wedge \cdots \wedge u_k \wedge v_1 \wedge \cdots \wedge v_\ell\| = \|u_1 \wedge \cdots \wedge u_k\| \cdot \|v_1 \wedge \cdots \wedge v_\ell\|.$$

<u>Pf</u> (a) Let v' be the orthogonal projection of v onto $(\mathrm{span}_{\mathbb{R}}(u_1, \dots, u_k))^{\perp}$.

That is, $v' = v - u$ where $u \in \mathrm{span}_{\mathbb{R}}(u_i)$,

$\langle v', u_i \rangle = 0$ for all i.

Then by Pythagoras, $\|v\|^2 = \|v'\|^2 + \|u\|^2$

and in particular, $\|v'\| \leq \|v\|$.

Hence $\quad \|u_1 \wedge \cdots \wedge u_k \wedge v\| = \|u_1 \wedge \cdots \wedge u_k \wedge (v' - u)\|$

$$\underset{\substack{u_1 \wedge \cdots \wedge u_k \wedge u = 0 \\ \text{by Cor 1}}}{=} \|u_1 \wedge \cdots \wedge u_k \wedge v'\| \underset{\text{by (b)}}{=} \|u_1 \wedge \cdots \wedge u_k\| \cdot \|v'\|$$

$$\leq \|u_1 \wedge \cdots \wedge u_k\| \cdot \|v\|$$

So the proof of (a) will be complete once we have (b).

Pf of (b): By Laplace identity:

$$\|u_1\wedge\cdots\wedge u_k\wedge v_1\wedge\cdots\wedge v_\ell\|^2 = \langle u_1\wedge\cdots\wedge v_\ell,\ u_1\wedge\cdots\wedge v_\ell\rangle$$

$$= \det\left(\begin{array}{c|c} \langle u_i, u_j\rangle & \overset{0}{\langle u_i, v_{j'}\rangle} \\ \hline \underset{0}{\langle v_{i'}, u_j\rangle} & \langle v_{i'}, v_{j'}\rangle \end{array}\right)\begin{array}{l} \}k \\ \}\ell \end{array} =$$

$i, j = 1, \ldots, k$

$i', j' = 1, \ldots, \ell$

$$= \det\big(\langle u_i, u_j\rangle\big)\cdot\det\big(\langle v_i, v_j\rangle\big) = \|u_1\wedge\cdots\wedge u_k\|^2\cdot\|v_1\wedge\cdots\wedge v_\ell\|^2$$

Take roots of both sides.

<u>Cor</u> (Hadamard inequality)

$$\left|\det\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}\right| \le \|v_1\|\cdots\|v_n\|$$

<u>Pf</u>: $\left|\det\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}\right| = \left|\det\begin{pmatrix} & \end{pmatrix}\right| \|e_{1,\ldots,n}\|$

$$= \|\det(v_1 \cdots v_n)\, e_1\wedge\cdots\wedge e_n\| \underset{\text{Prop }(*)}{=} \|v_1\wedge\cdots\wedge v_n\|$$

$\le \|v_1\| \cdot \ldots \cdot \|v_n\|.$

induction and part ⓐ of previous Cor.

<u>Cor</u> If $L_0 \subset L$ L a lattice, L_0 an additive subgroup of rank p, $L_0 = \mathrm{span}_{\mathbb{Z}}(u_1, \ldots, u_p)$.

Then $\mathrm{covol}(L_0) = \|u_1 \wedge \ldots \wedge u_p\|$.

<u>Rk</u> So decomposable elements $u_1 \wedge \ldots \wedge u_k$ carry two bits of information: $\mathrm{span}_{\mathbb{R}}(u_1, \ldots, u_k)$ which is encoded by $\mathbb{R} \cdot (u_1 \wedge \ldots \wedge u_k)$ (previous remark), and the covolume of $\mathrm{span}_{\mathbb{Z}}(u_1, \ldots, u_k)$ in $\mathrm{span}_{\mathbb{R}}(u_1, \ldots, u_k)$ encoded by $\|u_1 \wedge \ldots \wedge u_k\|$.

Pf: Assume first $w_1, \dots, w_p$ are orthogonal and lin. ind. ($\langle w_i, w_j \rangle = 0$ if $i \neq j$, $\|w_i\| \neq 0$).

Then by Laplace identity:

$$\|w_1 \wedge \dots \wedge w_p\|^2 = \langle w_1 \wedge \dots \wedge w_p, w_1 \wedge \dots \wedge w_p \rangle$$

$$= \det \left(\langle w_i, w_j \rangle \right) = \det \begin{pmatrix} \|w_1\|^2 & & 0 \\ & \ddots & \\ 0 & & \|w_p\|^2 \end{pmatrix} =$$

$$= \prod_i \|w_i\|^2 = \left(\mathrm{Vol}_V \left(\left\{ \sum_{i=1}^p a_i w_i : a_i \in [0,1] \right\} \right) \right)^2$$

$V = \mathrm{span}(w_i)$.

$$\mathrm{covol}(L_0) = \mathrm{Vol}_V \left(\left\{ \sum_{i=1}^p a_i v_i : a_i \in [0,1] \right\} \right).$$

Apply Gram-Schmidt to the v_i to form

an orthogonal collection $w_1, \dots, w_p$

$$w_1 = u_1, \quad w_2 = u_2 - \frac{\langle u_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$$

$$, \dots, \quad w_p = u_p - \sum_{i=1}^{p-1} \frac{\langle u_p, w_i\rangle}{\langle w_i, w_i\rangle} w_i$$

Then $\text{covol}(L_0) = \text{Vol}_V\left(\left\{\sum a_i u_i : a_i \in [0,1)\right\}\right)$

$$= \text{Vol}_V\left(\left\{\sum_{i=1}^{p} a_i w_i : a_i \in [0,1)\right\}\right) = \|w_1\| \cdots \|w_p\|.$$

box obtained from
parallelepiped by a lin. trans.
of det 1.

$$u_1 \wedge \cdots \wedge u_p = u_1 \wedge \cdots \wedge u_{p-1} \wedge (w_p + \text{lin. comb of } u_1, \dots, u_{p-1})$$

$$= u_1 \wedge \cdots \wedge u_{p-1} \wedge w_p = u_1 \wedge \cdots \wedge u_{p-2} \wedge w_{p-1} \wedge w_p$$

$$= \cdots = w_1 \wedge \cdots \wedge w_p.$$

<u>Cor:</u> Suppose $L_1 \subset L_2$ are both primitive additive subgroups of a lattice L

$V_1 = \mathrm{span}_{\mathbb{R}}(L_1)$ $\quad \pi : \mathbb{R}^n \longrightarrow V_1^{\perp}$

Then $\pi(L_2)$ is discrete in $V_1^{\perp}$ and

$\mathrm{covol}(\pi(L_2)) \cdot \mathrm{covol}(L_1) = \mathrm{covol}(L_2)$.

Pf Let $u_1, \dots, u_s$ be a basis for L_1.

We can complete L_1 to a basis of L_2

denote it by $u_1, \dots, u_s, v_{s+1}, \dots, v_r$.

$\|u_1 \wedge \dots \wedge u_s\| = \mathrm{covol}(L_1)$

$\pi(v) = v - u(v) \qquad u(v) \in V_1 = \mathrm{span}(L_1)$.

$$\mathrm{covol}(L_2) = \|u_1 \wedge \dots \wedge u_s \wedge v_{s+1} \wedge \dots \wedge v_r\|$$

$$= \|u_1 \wedge \dots \wedge u_s \wedge (\pi(v_{s+1}) + u(v_{s+1})) \wedge \dots \wedge (\pi(v_r) + u(v_r))\|$$

$$= \|u_1 \wedge \dots \wedge u_s \wedge \pi(v_{s+1}) \wedge \dots \wedge \pi(v_r)\|$$

$$= \|u_1 \wedge \dots \wedge u_s\| \cdot \|\pi(v_{s+1}) \wedge \dots \wedge \pi(v_r)\|$$

$$= \mathrm{covol}(L_1) \cdot \mathrm{covol}(\pi(L_2))$$

Let $L_1, L_2 \subset L$ be two additive subgroups of a lattice. Denote by $L_1 + L_2$ the group generated by L_1, L_2, i.e.

$$L_1 + L_2 = \{u+v : u \in L_1, v \in L_2\}.$$

If $L_1 = \mathrm{span}_{\mathbb{Z}}(u_1, \dots, u_s)$

$L_2 = \mathrm{span}_{\mathbb{Z}}(v_1, \dots, v_r)$

then $L_1 + L_2 = \mathrm{span}_{\mathbb{Z}}(u_1, \dots, u_s, v_1, \dots, v_r)$

<u>Prop</u> Suppose L_1, L_2 are primitive. Then

$$(*) \quad \mathrm{covol}(L_1 + L_2) \cdot \mathrm{covol}(L_1 \cap L_2) \leq \mathrm{covol}(L_1) \cdot \mathrm{covol}(L_2).$$

<u>Pf</u>: Suppose first $L_1 \cap L_2 = \{0\}$.

<u>Claim:</u> $u_1, \dots, u_s, v_1, \dots, v_r$ are lin. ind.

Because. If $\sum a_i u_i + \sum b_j v_j = 0$

Then $\sum a_i u_i = -\sum b_j v_j \in L_1 \cap L_2$.

So $\sum a_i u_i = 0 = \sum b_j v_j$

$\Longrightarrow a_1 = \cdots = a_s = b_1 = \cdots = b_r = 0$. Proves claim.

Hence $\operatorname{covol}(L_1+L_2)\cdot\operatorname{covol}(L_1\cap L_2)$

$$= \operatorname{covol}(L_1+L_2) = \|u_1\wedge\cdots\wedge u_s\wedge v_1\wedge\cdots\wedge v_r\|$$

$$\le \|u_1\wedge\cdots\wedge u_s\|\cdot\|v_1\wedge\cdots\wedge v_r\| = \operatorname{covol}(L_1)\cdot\operatorname{covol}(L_2).$$

In the general case, we claim:

there is a primitive r-tuple $w_1,\dots,w_r$ s.t. $\operatorname{span}_{\mathbb{Z}}(w_1,\dots,w_r) = L_1\cap L_2$

and $\operatorname{span}_{\mathbb{Z}}(w_1,\dots,w_r) = L_2\cap\operatorname{span}_{\mathbb{R}}(L_i)$

$i=1,2$.

Assuming claim, $w_1,\dots,w_r$ is primitive in both L_1 and L_2.

So can find u_i', v_j' s.t.

$$L_1 = \operatorname{span}_{\mathbb{Z}}(w_1,\dots,w_r,u_1',\dots,u_{s'}')$$

$$L_2 = \operatorname{span}_{\mathbb{Z}}(w_1,\dots,w_r,v_1',\dots,v_{r'}')$$

$L_1 + L_2 = \mathrm{span}_{\mathbb{Z}}(w_1, \dots, w_r, u_1', \dots, u_s', v_1', \dots, v_{r'}')$

Apply previous cor with $L_0 = L_1 \cap L_2$.

$\pi : \mathbb{R}^n \longrightarrow \mathrm{span}(L_0)^{\perp}$

This gives $\mathrm{covol}(L_i) =$ $\quad i=1,2$

$$= \mathrm{covol}(\pi(L_i)) \cdot \mathrm{covol}(L_0)$$

$$\mathrm{covol}(L_1 + L_2) = \mathrm{covol}(\pi(L_1 + L_2)) \cdot \mathrm{covol}(L_0).$$

From previous case,

$$\mathrm{covol}(\pi(L_1 + L_2)) \leq \mathrm{covol}(\pi(L_1)) \cdot \mathrm{covol}(\pi(L_2))$$

Multiplying both sides by $\mathrm{covol}(L_0)^2$ get $(*)$.

Pf of claim Let $L = A(\mathbb{Z}^n)$ for $A \in GL_n(\mathbb{R})$.

Multiplying by A^{-1}, can assume $L = \mathbb{Z}^n$.

Write $V_i = \mathrm{span}(L_i)$, $i=1,2$, and $V_0 = V_1 \cap V_2$.

We say a subspace of $\mathbb{R}^n$ is <u>defined over $\mathbb{Q}$</u> if it is the nullset of a system of linear equations with rational coefficients.

Then V_i is the span (over $\mathbb{R}$) of vectors in $\mathbb{Z}^n$ and hence each V_i is defined over $\mathbb{Q}$. Therefore so is V_0. Hence $V_0 \cap \mathbb{Q}^n$ contains s linearly independent vectors, where $s = \dim V_0$, and hence the same is true of $L_0 = V_0 \cap \mathbb{Z}^n$. Clearly L_0 is discrete and hence L_0 is a lattice in V_0.

Let $w_1, \dots, w_r$ s.t. $L_0 = \operatorname{span}_{\mathbb{Z}}(w_1, \dots, w_r)$.

Since L_1, L_2 are both primitive, $\mathbb{Z}^n \cap V_i = L_i$ and hence

$$L_0 = V_0 \cap \mathbb{Z}^n = (V_1 \cap V_2) \cap \mathbb{Z}^n = (V_1 \cap \mathbb{Z}^n) \cap (V_2 \cap \mathbb{Z}^n)$$
$$= L_1 \cap L_2.$$

This proves claim.