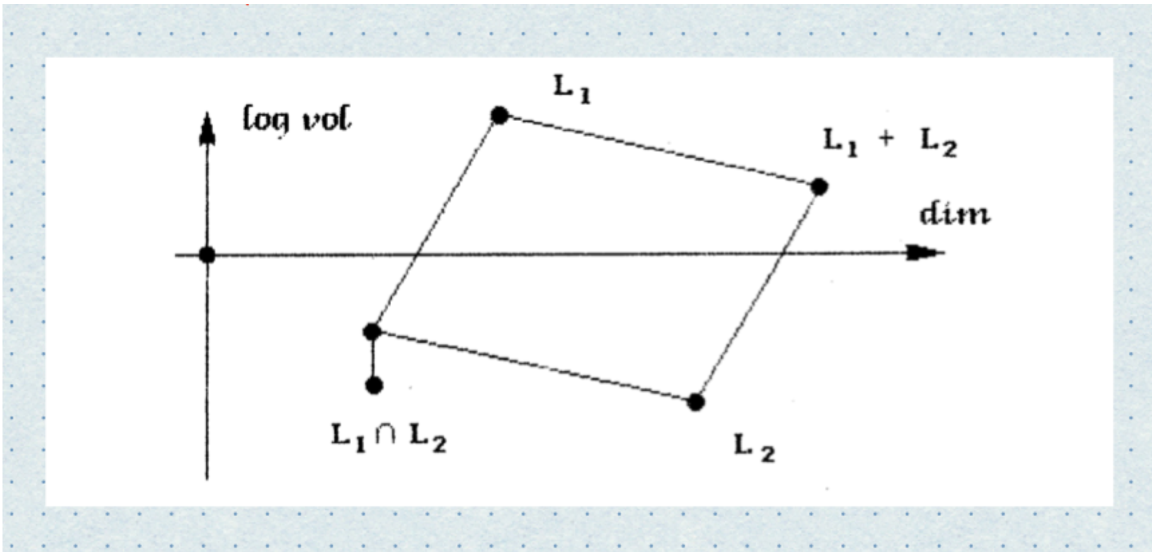
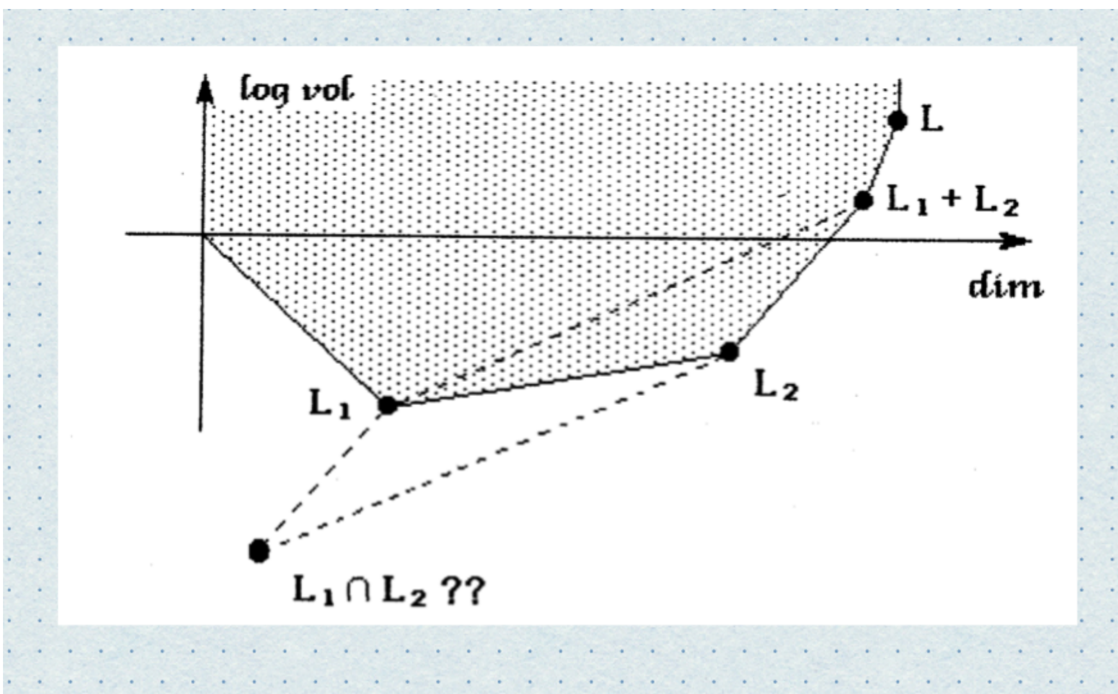
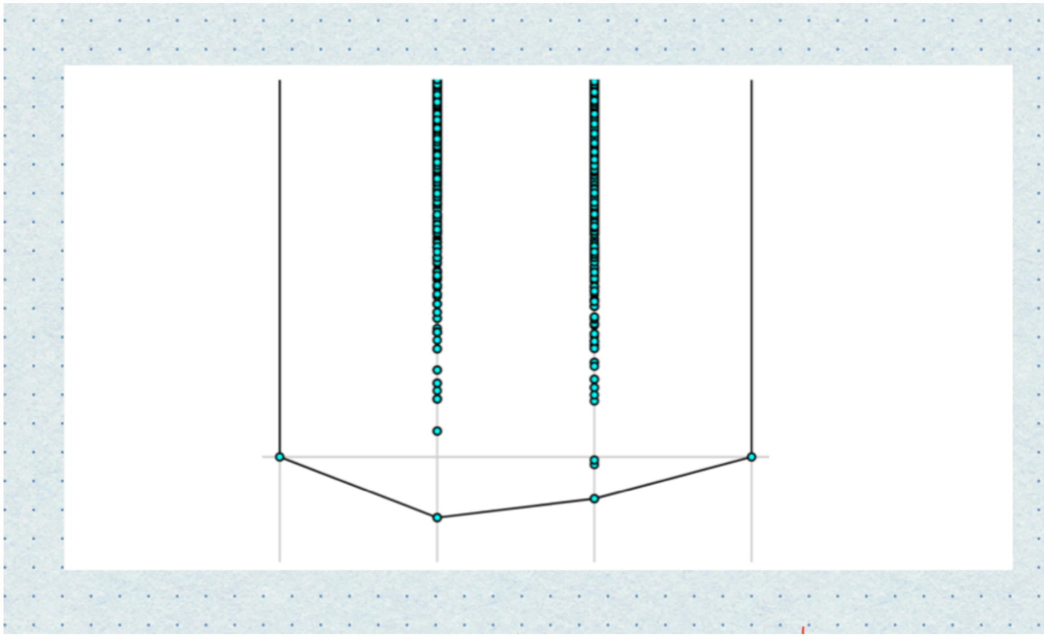




Pictures from D. Grayson, Reduction theory via semistability, '84.



Picture: Casselman, Stability of lattices and the partition of arithmetic quotients, '04

Geometry of Numbers Lecture 5

Recall: $G_n = \bigoplus_{p=0}^{n} \mathbb{R}^n_p$

$\mathbb{R}^n_p = \text{span}\{ e_\sigma : \sigma = (1 \le i_1 < \cdots < i_p \le n)\}$

Notation: $\mathbb{R}^n_p = \Lambda^p \mathbb{R}^n$

Last week I "proved" that $\forall u, v \in G_n$

$\|u \wedge v\| \le \|u\| \cdot \|v\|$.

In the notes you will find a corrected proof in the case u, v are decomposable, i.e.

$$u = u_1 \wedge \cdots \wedge u_p \quad , v = v_1 \wedge \cdots \wedge v_q$$

$$u_i, v_j \in \mathbb{R}^n.$$

Recall from previous lecture:

If $L_0 \subset L$ is a subgroup of rank r, i.e. $L_0 = \mathrm{span}_{\mathbb{Z}}(u_1, \dots, u_r)$ u_i lin. ind.

then $\mathrm{covol}(L_0) = \| u_1 \wedge \cdots \wedge u_r \|$

For $L_1, L_2 \subset L$ two subgroups

$$L_1 + L_2 = \{ l_1 + l_2 : l_i \in L_i \} = \mathrm{span}_{\mathbb{Z}}(L_1 \cup L_2).$$

<u>Prop</u> $\forall L_1, L_2$ primitive

$$\mathrm{covol}(L_1 + L_2) \cdot \mathrm{covol}(L_1 \cap L_2) \leq \mathrm{covol}(L_1) \cdot \mathrm{covol}(L_2).$$

complete proof in lecture notes (including a proof of a claim not given in the lecture).

$$\alpha_i(L) = \inf \left\{ \mathrm{covol}(L_0) : L_0 \subset L \text{ a primitive subgroup of rank } i \right\}.$$

Problems with our "measurements" of geometry of a lattice.

We had λ_i (successive minima)
κ_i (lengths of the basis given by Korkine-Zolotarev process).
α_i

Notation: $A \asymp B$ means $\exists C > 0$ (ind. of L, depends on n), such that for all $L < \mathbb{R}^n$,

$$\frac{1}{C} B(L) \leq A(L) \leq C B(L)$$

$\lambda_i \asymp \kappa_i$

$\alpha_i \asymp \lambda_1 \cdots \lambda_i$

For $\lambda_i, \kappa_i, \alpha_i$ we have minimizers ($\|v_i\| = \lambda_i$, $\|v_i'\| = \kappa_i$, $\mathrm{covol}(L_i) = \alpha_i$)

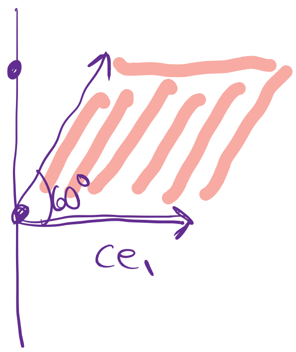
Problems
* minimizers not well-defined (can have ties).
* Hard to find (many "near-ties").
* v_i's realizing λ_i's not a basis for L.
* α_i not necessarily realized by $\mathrm{span}_{\mathbb{Z}}(v_1, \ldots, v_i)$, if $\|v_i\| = \lambda_i$.

Solution: <u>Harder-Narasimhan filtration</u>

<u>Examples</u>: $n=3$ $\quad L_0 = L_1 \oplus \mathbb{Z} e_3 \quad L_1 \subset \mathbb{R}^2$

L_1 is hexagonal lattice, rescaled to have covolume 1.

$c\sin(60°) = c\frac{\sqrt{3}}{2}$



$c^2 \frac{\sqrt{3}}{2} = 1$

$c = \sqrt{\frac{2}{\sqrt{3}}} = 1.074\ldots$

$\alpha_1 = \lambda_1 = 1$, shortest vectors are $\pm e_3$.

Let's compute α_2. Recall that if $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$,

$$u \wedge v = \begin{pmatrix} u_1 v_2 - u_2 v_1 \\ u_3 v_1 - u_1 v_3 \\ u_2 v_3 - u_3 v_2 \end{pmatrix}$$

$$\|u \wedge v\| \begin{cases} \geq |u_1 v_2 - u_2 v_1| \geq 1 & \text{if } u_1 v_2 - u_2 v_1 \neq 0 \\ = \|\underbrace{(k_1 w + \ell_1 e_3) \wedge (k_2 w + \ell_2 e_3)}_{\text{for } w \in L_1}\| & \text{if } u_1 v_2 - u_2 v_1 = 0 \end{cases}$$

$$\| \underbrace{(k_1 \ell_2 - k_2 \ell_1)}_{\text{nonzero integer}} w \wedge e_3 \| \geq \|w \wedge e_3\| = \|w\| \cdot \|e_3\| \geq$$

$$= \|w\| \geq 1.074\ldots$$

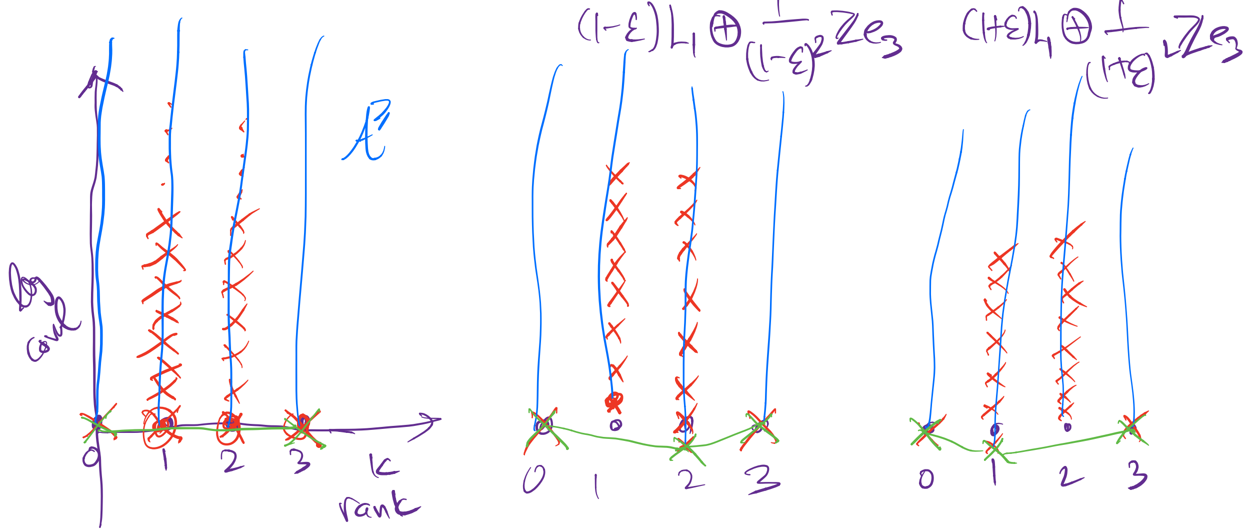
so $\alpha_2 = 1$, realized by L_1.

So λ_1 realized by e_1
α_2 realized by L_1 $e_1 \notin L_1$.

Graph the numbers $(k, \log \operatorname{covol}(L_0))$
where $k \in \{0, \dots, n\}$, $L_0 \subset L$ has rank k.

L_0 $\varepsilon > 0$ small

$(1-\varepsilon)L_1 \oplus \frac{1}{(1-\varepsilon)^2}\mathbb{Z}e_3$ $(1+\varepsilon)L_1 \oplus \frac{1}{(1+\varepsilon)^2}\mathbb{Z}e_3$

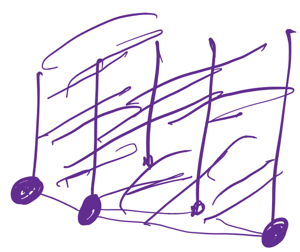


X vertices of profile

Define $\mathcal{A} = \{(k, \log \operatorname{covol}(L_0)) : L_0 \subset L,\ L_0 \text{ has rank } k,\ k \in \{0, \dots, n\}\}$.

$\tilde{\mathcal{A}} = \{(k, y) : k \in \{0, \dots, n\},\ y \geq \log \alpha_k(L)\}$

The extreme points on the bottom of $\mathrm{conv}(A^{\geq})$ form a polygonal line called the <u>profile of L</u>.



<u>Thm</u> ① The minima α_i realizing the heights of vertices of the profile are of the form $\log \mathrm{covol}(L_i)$, and the L_i are <u>unique</u> (i.e., cannot have $L_i, L_i' \subset L$, $L_i \neq L_i'$, with the same covol, realizing α_i).

② The sublattices L_i for which $(\mathrm{rank}(L_i), \log \mathrm{covol}(L_i))$ is a vertex of the profile of L, form a <u>flag</u>, i.e. are nested:

$$\{0\} = L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_k \subsetneq L_{k+1} = L \qquad (*)$$

$(\mathrm{rank}(L_i), \log\mathrm{covol}(L_i))$ are the vertices of the profile. The collection $(\star)$ is called the <u>Harder–Narasimhan filtration</u>, and k is its <u>length</u>.

(3) For each $i \in \{1, \dots, k\}$

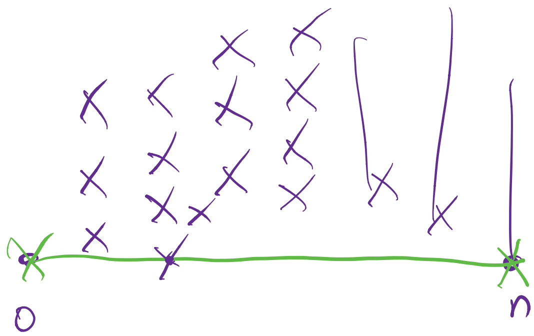
$\{0\} = L_0 \subsetneq \cdots \subsetneq L_i$ is the HN filtration of L_i, and if $\pi : \mathbb{R}^n \to \mathrm{span}(L_i)^\perp$ is the orthogonal projection, then

$\{0\} = \pi(L_i) \subsetneq \pi(L_{i+1}) \subsetneq \cdots \subsetneq \pi(L)$ is the HN filtration of $\pi(L)$.

(4) If $k = 0$ we say L is <u>stable</u>.

For each i, L_{i+1}/L_i (considered as a lattice in $\mathrm{span}\,\pi(L_{i+1})$) is stable.

$$1 \to L_0 \to L \to L/L_0 \to 1$$



$$\mathrm{covol}(L_1\cap L_2)\cdot \mathrm{covol}(L_1+L_2) \le \mathrm{covol}(L_1)\,\mathrm{covol}(L_2)$$

$$\Downarrow$$

$$\log\mathrm{covol}(L_1\cap L_2)+\log\mathrm{covol}(L_1+L_2) \le \log\mathrm{covol}(L_1)+\log\mathrm{covol}(L_2).$$

$$y_\cap \quad + \quad y_+ \quad \le \quad y_1+y_2$$

this inequality has an interpretations as a <u>parallelogram diagram</u>:

L_1, L_2 are subgroups of L of rank k, l.

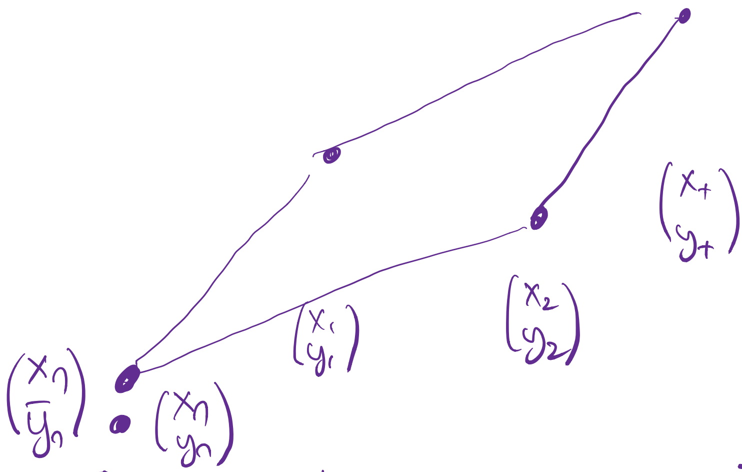
Suppose $\mathrm{rank}(L_1+L_2)=m$

Then $\mathrm{rank}(L_1\cap L_2)=k+l-m$

$k=x_1 \quad l=x_2 \quad m=x_+ \quad k+l-m=x_\cap$

$y_i=\log\mathrm{covol}(L_i) \;, i=1,2$

$y_+ = \log \operatorname{covol}(L_1 + L_2)$ $\quad y_\cap = \log \operatorname{covol}(L_1 \cap L_2)$.



$\bar{y}_\cap = y_1 + y_2 - y_+$ $\qquad y_\cap \le \bar{y}_\cap$

$$\begin{pmatrix} x_\cap \\ \bar{y}_\cap \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} - \begin{pmatrix} x_+ \\ y_+ \end{pmatrix}$$

Pf of thm 1 Suppose $(k, \log \alpha_k)$ is a vertex of the profile, $\alpha_k = \operatorname{covol}(L_1)$
$= \operatorname{covol}(L_2)$

$\operatorname{rank}(L_1) = \operatorname{rank}(L_2) = k$.

In the parallelogram diagram we get

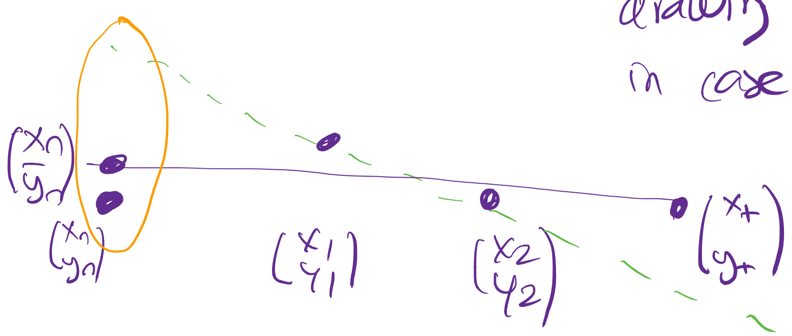
$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, so line connecting $\begin{pmatrix} x_\cap \\ y_\cap \end{pmatrix}$ to $\begin{pmatrix} x_+ \\ y_+ \end{pmatrix}$ is below $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$. Contradicts the fact that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ is an extreme pt of the profile, unless σ is degenerate, i.e.

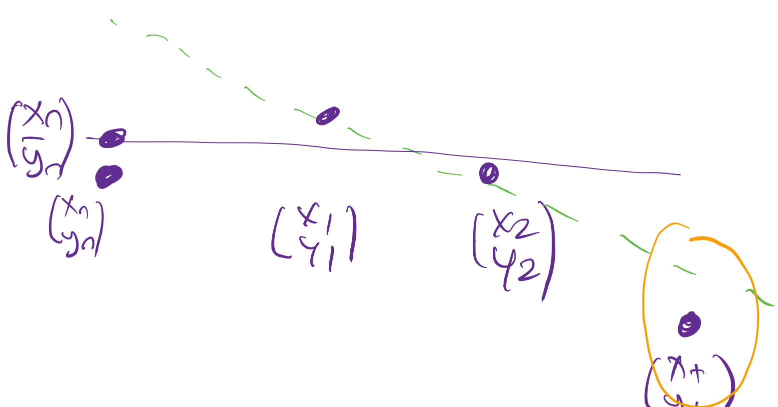
$x_\cap = x_1 \cap x_2 \Rightarrow \dim(L_1 \cap L_2) = \dim(L_1) = \dim(L_2)$

$\Rightarrow L_1 \cap L_2 = L_1 = L_2.$

2 $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ are both extreme pts. of profile, suppose $x_1 < x_2$.

drawing the contradiction in case $\begin{pmatrix} x_+ \\ y_+ \end{pmatrix}$ above the line through $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$





not allowed to have points of the form $(m, \log \operatorname{count}(L_0))$, $\operatorname{rank}(L_0) = m$, below green line:

In both cases have a contradiction, unless

$x_n = x_1 \Rightarrow \mathrm{rank}(L_1 \cap L_2) = \mathrm{rank}(L_1)$

$\Rightarrow \quad L_1 \cap L_2 = L_1 \Rightarrow L_1 \subset L_2.$

③ $\mathcal{A}(L) = \left\{ (k, \log \mathrm{covol}(L_0)) : \begin{array}{l} L_0 \subset L \\ \mathrm{rank}(L_0) = k \end{array} \right\}$

$\mathcal{A}(L_i) \subset \mathcal{A}(L)$, for any $L_i \subset L$.

$\mathcal{A}^{\geq}(L_i)$ lies above $\mathcal{A}^{\geq}(L)$.

pts in the profile of L, with $\mathrm{rank}(L_0) \leq i$,

also belong to $\mathcal{A}^{\geq}(L_i)$, by ②.

So profile of L_i is the profile of L,

restricted to x coordinate in $\{0, \dots, \mathrm{rank}(L_i)\}$.

③ second statement
④
$\Big\}$ ex.

The space of lattices

Recall: There is a bijection

$$\left\{\begin{matrix}\text{lattices in} \\ \mathbb{R}^n\end{matrix}\right\} \longleftrightarrow GL_n(\mathbb{R})/GL_n(\mathbb{Z})$$

$GL_n(\mathbb{R})$: $n\times n$ inv. real matrices

$GL_n(\mathbb{Z})$: $n\times n$ integer matrices with $\det = \pm 1$

$$g\mathbb{Z}^n \longleftarrow g\,GL_n(\mathbb{Z}), \quad g \in GL_n(\mathbb{R})$$

This is a bijection because $GL_n(\mathbb{R})$ acts transitively on all lattice, $GL_n(\mathbb{Z})$ is stabilizer of $\mathbb{Z}^n$.

$$\left\{\begin{matrix}\text{lattices} \\ \text{of covol} = 1\end{matrix}\right\} \longleftrightarrow SL_n(\mathbb{R})/SL_n(\mathbb{Z})$$

$SL_n(\mathbb{R})$: $n\times n$ real matrices of $\det = 1$

$SL_n(\mathbb{Z})$: $n\times n$ integer matrices of $\det = 1$

(check $SL_n(\mathbb{R})$ transitive on lattices of covol $= 1$ and $SL_n(\mathbb{Z})$ stabilizer of $\mathbb{Z}^n$).

$$Cl(\mathbb{R}^n) = \left\{\begin{matrix}\text{closed subsets} \\ \text{of } \mathbb{R}^n\end{matrix}\right\}$$

$\left\{ \begin{matrix} \text{lattices} \\ \text{in } \mathbb{R}^n \end{matrix} \right\} \subset Cl(\mathbb{R}^n)$

On $Cl(\mathbb{R}^n)$, define the Chabauty-Fell metric by

$$D(X,Y) = \inf \left\{ \{1\} \cup \left\{ \varepsilon \in (0,1) : \begin{matrix} \forall x \in B(0,\tfrac{1}{\varepsilon}) \cap X, \exists y \in Y \text{ with } \|x-y\| < \varepsilon \\ \forall y \in B(0,\tfrac{1}{\varepsilon}) \cap Y, \exists x \in X \text{ with } \|x-y\| < \varepsilon \end{matrix} \right\} \right\}$$

<u>Ex</u> 1. this is a metric

2. $Cl(\mathbb{R}^n)$ is compact

<u>Prop</u> Let $L, L_1, L_2, \ldots$ be lattices in $\mathbb{R}^n$.

The following are equivalent: (a) $L_j \xrightarrow[j\to\infty]{} L$ (w.r.t. D)

(b) (i) $\forall \ell \in L\ \exists \ell_j \in L_j$ s.t. $\ell_j \xrightarrow[j\to\infty]{} \ell$

(ii) If $j_k \nearrow \infty$, $\ell_{j_k} \in L_{j_k}$ s.t. $\ell_\infty = \lim_{k\to\infty} \ell_{j_k}$ exists,

then $\ell_\infty \in L$.

(c) For any basis $v_1, \ldots, v_n$ of L $\exists$ bases $v_1^{(j)}, \ldots, v_n^{(j)}$ of L_j s.t. for $i = 1, \ldots, n$,

$v_i^{(j)} \xrightarrow[j\to\infty]{} v_i$.

(d) Writing $L_j = g_j \mathbb{Z}^n$ $g_j \in GL_n(\mathbb{R})$

$L = g\mathbb{Z}^n$ $g \in GL_n(\mathbb{R})$

$\exists\, \gamma_j \in GL_n(\mathbb{Z})$ s.t. $g_j \gamma_j \xrightarrow[j\to\infty]{} g$.

(convergence in $GL_n(\mathbb{R}) \subset \mathbb{R}^{n^2}$).

The functions $L \longmapsto \lambda_1(L)$

$L \longmapsto \operatorname{covol}(L)$

are both continuous w.r.t. D, i.e

$$L_j \to L \Rightarrow \begin{cases} \lambda_1(L_j) \longrightarrow \lambda_1(L) \\ \operatorname{covol}(L_j) \longrightarrow \operatorname{covol}(L) \end{cases}$$

Pf (a) $\Rightarrow$ (b) $D(L_j, L) \xrightarrow[j\to\infty]{} 0$

$\varepsilon_j = 2D(L_j, L) \longrightarrow 0$

(i) Given $\ell \in L$, choose j_0 so that

$\frac{1}{\varepsilon_j} > \|\ell\|$ $\forall j \geq j_0$. Then by $D(L, L_j) < \varepsilon_j$

$\exists\, \ell_j \in L_j$ s.t. $\|\ell - \ell_j\| < \varepsilon_j \longrightarrow 0$

so $\ell_j \to \ell$.

(ii) Suppose $l_{j_k} \in L_{j_k}$, $l_{j_k} \to l_\infty$

Let j_0 s.t. $\frac{1}{\varepsilon_j} > \|l_\infty\| + 1 \quad \forall j \geq j_0$,

and $\|l_j - l_\infty\| < 1 \quad \forall j \geq j_0$

By def'n of D $\exists\ \bar{l}_{j_k} \in L$ s.t. $\|l_{j_k} - \bar{l}_{j_k}\| < \varepsilon_{j_k} \to 0$

$\bar{l}_{j_k} \xrightarrow[k\to\infty]{} l_\infty \implies l_\infty \in L$.

(L closed)

(b) $\Rightarrow$ (c) We first show (b) $\Rightarrow \lambda_1(L_j) \to \lambda_1(L)$.

Let $v_j \in L_j$ s.t. $v_j \to v$, $\|v\| = \lambda_1(L)$

$\lambda_1(L_j) \leq \|v_j\|$

$\limsup_{j\to\infty} \lambda_1(L_j) \leq \limsup_{j\to\infty} \|v_j\| = \|v\| = \lambda_1(L)$.

Let $c = \liminf_{j\to\infty} \lambda_1(L_j)$, suppose by contr.

that $c < \lambda_1(L)$. If $c > 0$ $\exists\ j_k \nearrow \infty$

and $v_{j_k} \in L_{j_k}$ s.t. $\|v_{j_k}\| = \lambda_1(L_{j_k})$

* Passing to a further subsequence,

$v_{j_k} \to v_\infty \in L$ (by ⓑ).

$c = \lim_{k\to\infty} \|v_{j_k}\| = \|v_\infty\| \geq \lambda_1(L)$ contr.

If $c=0$, then for large enough k

L_{j_k} contains a vector v_{j_k} of length

$< \frac{\lambda_1(L)}{3}$. Multiplying by an integer, set

$u_{j_k} \in L_{j_k}$ of length in $\left[\frac{\lambda_1(L)}{3}, \frac{2\lambda_1(L)}{3}\right]$

Passing to a further subsequence,

$u_{j_k} \to w \in L$, $\|w\| \in \left[\frac{\lambda_1(L)}{3}, \frac{2}{3}\lambda_1(L)\right]$

contradiction.

Continuing with ⓑ ⇒ ⓒ. By ⓑ $\exists\, v_i^{(j)} \in L_j$

s.t. $v_i^{(j)} \to v_i$.

Let $\bar{L}_j = \mathrm{span}_{\mathbb{Z}}(v_1^{(j)}, \dots, v_n^{(j)}) \subset L_j$.

Conclusion is satisfied for $\bar{L}_j$ instead of L_j:

Also $\operatorname{covol}(L_j) = |\det(v_1^{(j)} \cdots v_n^{(j)})| \xrightarrow[j\to\infty]{} |\det(v_1 \cdots v_n)| = \operatorname{covol}(L)$

Suppose by contradiction that along an infinite subsequence of j, $\bar{L}_j \neq L_j$.

(From now on, pass to subsequences freely).

$[L_j : \bar{L}_j] > 1$. By cont. of λ_1,

$\lambda_1(L_j)$ is bounded below (ind. of j).

So $\exists \eta > 0$ $\exists C > 0$ s.t.

$$0 < \eta \leq \lambda_1^n(L_j) \leq \lambda_1(L_j) \cdots \lambda_n(L_j) \overset{\text{Mink. 2nd thm}}{\leq} C \operatorname{covol}(L_j)$$
$$\leq C \operatorname{covol}(\bar{L}_j) \to C \operatorname{covol}(L).$$

Passing to a subsequence, $\operatorname{covol}(L_j)$ converges to some positive number, and

$\operatorname{covol}(\bar{L}_j) = [L_j : \bar{L}_j] \operatorname{covol}(L_j)$

$[L_j : \bar{L}_j] = M$ (along a subsequence).

By a result proved in lecture 1, there is a basis $u_1^{(j)}, \ldots, u_n^{(j)}$ of L_j, s.t.

$$v_1^{(j)} = a_{11} u_1^{(j)}$$
$$v_2^{(j)} = a_{21} u_1^{(j)} + a_{22} u_2^{(j)}$$

$\vdots$

$$V_n^{(j)} = a_{n1} V_1^{(j)} + \cdots + a_{nn} u_n^{(j)}$$

a_{kl} depend on j, and satisfy:

$a_{kl} \in \mathbb{Z}$, $\quad a_{ii} \in \mathbb{N}$ $\quad a_{11} \cdot \ldots \cdot a_{nn} = M$

$|a_{kl}| \leq a_{ll}$

Passing to a subsequence, can assume a_{kl} are independent of j.

Let i_0 be the small index so that $u_{i_0}^{(j)} \notin L_j$

i.e. $a_{11} = \cdots = a_{i_0-1\, i_0-1} = 1$, $a_{i_0 i_0} > 1$.

Write $m = a_{i_0 i_0} > 1$.

Then for $i = 1, \ldots, i_0 - 1$, $u_i^{(j)} \in \operatorname{span}_{\mathbb{Z}}(v_1^{(j)}, \ldots, v_i^{(j)})$

and $u_{i_0}^{(j)} = \sum_{i \leq i_0} b_i v_i^{(j)}$ with

$b_i \in \frac{1}{m}\mathbb{Z}$, $\quad b_{i_0} = \frac{1}{m}$.

Thus $u_{i_0}^{(j)} \notin L_j$.

Passing to a subsequence $u_{i_0}^{(j)} \to u \in L$ (by (5))

and $u = \sum_{i \le i_0} b_i v_i$ $\quad b_{i_0 i_0} = \frac{1}{m} \notin \mathbb{Z}$.

contradiction to the fact $v_1, \dots, v_n$ are a basis of L.

(c) $\Rightarrow$ (d) Let $L_j = g_j \mathbb{Z}^n$ $\quad L = g\mathbb{Z}^n$

$$L_j = \begin{pmatrix} | & & | \\ v_1^{(j)} & \cdots & v_n^{(j)} \\ | & & | \end{pmatrix} \mathbb{Z}^n = h_j \mathbb{Z}^n$$

$\exists\, \tau_j \in GL_n(\mathbb{Z})$ s.t. $g_j \tau_j = h_j$

$$h_j \to \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} = h \qquad h\mathbb{Z}^n = L$$

$h = g\tau$ for some $\tau \in GL_n(\mathbb{Z})$

$r_j = \tau_j \tau^{-1}$

$g_j r_j = g_j \tau_j \tau^{-1} \longrightarrow h\tau^{-1} = g$ proving (d).

(d) $\Rightarrow$ (a) Denote, for $g \in GL_n(\mathbb{R})$,

$$\|g\|_{op} = \sup_{\substack{x \in \mathbb{R}^n \\ \|x\|=1}} \frac{\|gx\|}{\|x\|}.$$

Given $\varepsilon > 0$, let j_0 be large enough s.t.

for all $j \geq j_0$, $\|g_j \gamma_j g^{-1} - Id\|_{op} < \varepsilon^2$

and $\|g \gamma_j^{-1} g_j^{-1} - Id\|_{op} < \varepsilon^2$.

The mapping $g_j \gamma_j g^{-1}$ maps $L = g\mathbb{Z}^n$ to $L_j = g_j \mathbb{Z}^n$

$g \gamma_j^{-1} g_j^{-1}$ " L_j " L.

For any $x \in L \cap B(0, \frac{1}{\varepsilon})$, then $g_j \gamma_j g^{-1} x \in L_j$

and $\|g_j \gamma_j g^{-1} x - x\| < \|x\| \cdot \varepsilon^2 \leq \varepsilon$.

Similarly $x \in L_j \cap B(0, \frac{1}{\varepsilon})$,

$\|g \gamma_j^{-1} g_j^{-1} x - x\| < \varepsilon$

$D(L_j, L) < \varepsilon$.

Remarks: (i) (a) $\Longleftrightarrow$ (b) works for any sequence in $Cl(\mathbb{R}^n)$. (ex.)

(b) The top. induced by D is the quotient top. on

$$GL_n(\mathbb{R}) / GL_n(\mathbb{Z}) .$$

(smallest top. on $GL_n(\mathbb{R})/GL_n(\mathbb{Z})$ making the proj. map cts.).