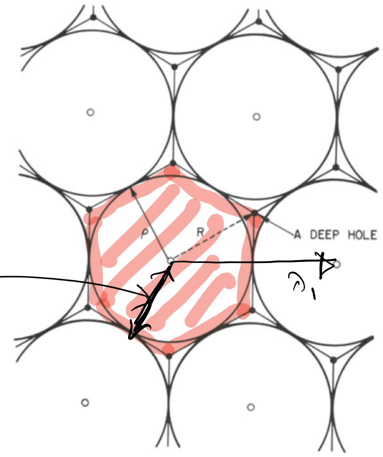
$\frac{\lambda_1(L)}{2}$



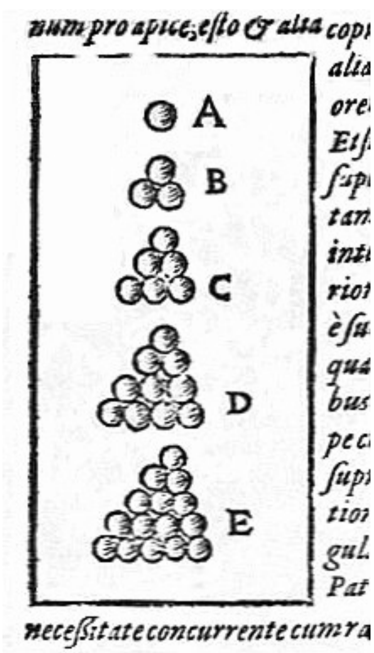
λ_1

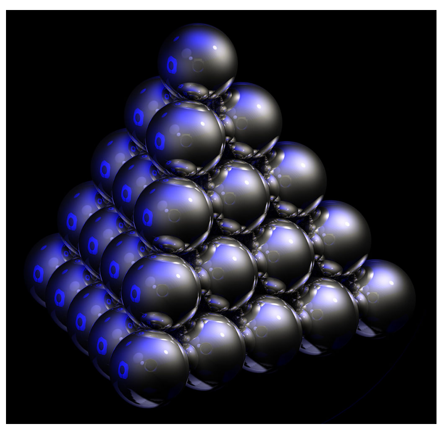
$V_n = \mathrm{Vol}(B(0,1))$

(d)

Figure 1.3 (cont.)

$$= \frac{\text{volume of one sphere}}{\text{volume of fundamental region}} = \frac{\left(\frac{\lambda_1(L)}{2}\right)^n \cdot V_n}{\mathrm{covol}(L)}$$





From "the six-cornered snowflake", Kepler 1611 (based on correspondence with Harriot).

dim n	lattice	proved optimal by (year)	optimal among all packings?
2	A2 (hexagonal lattice)	Lagrange 1773	yes Fejes-Toth 1943
3	A3=D3	Gauss 1831	yes Hale 1998, 2015
4	D4	Korkine-Zolotarev 1877	?
5	D5	Korkine-Zolotarev 1877	?
6	E6	Blichfeldt 1934 (following Voronoi 1908)	?
7	E7	Blichfeldt 1934 (following Voronoi 1908)	?
8	E8	Blichfeldt 1934	yes Viazovska 2016
24	Leech lattice	Cohn-Kumar 2004	yes Cohn-Kumar-Miller-Radchenko-Viazovska 2017

Lattices Lecture 6

Reminders from Lecture 5

On $Cl(\mathbb{R}^n)$, define the Chabauty-Fell metric by

$$D(X,Y) = \inf\left\{\{1\} \cup \left\{\varepsilon \in (0,1) : \begin{array}{l} \forall x \in B(0,\tfrac{1}{\varepsilon}) \cap X, \exists y \in Y \text{ with } \|x-y\| < \varepsilon \\ \forall y \in B(0,\tfrac{1}{\varepsilon}) \cap Y, \exists x \in X \text{ with } \|x-y\| < \varepsilon \end{array}\right\}\right\}$$

Prop Let $L, L_1, L_2, \ldots$ be lattices in $\mathbb{R}^n$.

The following are equivalent:

(a) $L_j \xrightarrow[j\to\infty]{} L$ (w.r.t. D)

(b) (i) $\forall \ell \in L \; \exists \ell_j \in L_j$ s.t. $\ell_j \xrightarrow[j\to\infty]{} \ell$

(ii) If $j_k \nearrow \infty$, $\ell_{j_k} \in L_{j_k}$ s.t. $\ell_\infty = \lim_{k\to\infty} \ell_{j_k}$ exists,

then $\ell_\infty \in L$.

(c) For any basis $v_1, \ldots, v_n$ of L $\exists$ bases

$v_1^{(j)}, \ldots, v_n^{(j)}$ ~~of~~ of L_j s.t. for $i = 1, \ldots, n$,

$$v_i^{(j)} \xrightarrow[j\to\infty]{} v_i .$$

(d) Writing $L_j = g_j \mathbb{Z}^n \quad g_j \in GL_n(\mathbb{R})$

$$L = g\mathbb{Z}^n \quad g \in GL_n(\mathbb{R})$$

$$\exists\ \gamma_j \in GL_n(\mathbb{Z}) \text{ s.t. } g_j\gamma_j \xrightarrow[j\to\infty]{} g .$$

(convergence in $GL_n(\mathbb{R}) \subset \mathbb{R}^{n^2}$).

The functions $L \longmapsto \lambda_1(L)$

$$L \longmapsto \operatorname{covol}(L)$$

are both continuous w.r.t. D.

$\overline{\mathcal{X}}_n \overset{\text{def}}{=} \{\text{lattices in } \mathbb{R}^n\} \xleftrightarrow{\text{bijection}} GL_n(\mathbb{R})/GL_n(\mathbb{Z})$

$\mathcal{X}_n \overset{\text{def}}{=} \left\{\begin{array}{l}\text{lattices in } \mathbb{R}^n \\ \text{of covolume one}\end{array}\right\} \longleftrightarrow SL_n(\mathbb{R})/SL_n(\mathbb{Z})$

$$g\mathbb{Z}^n \longleftarrow g\Gamma \quad \left(\Gamma = SL_n(\mathbb{Z}) \text{ or } GL_n(\mathbb{Z})\right)$$

Def: We say that $L_j \to \infty$ (L_j diverges in $\bar{\mathcal{X}}_n$) if the sequence $\{L_j\}$ has no convergent subsequence (w.r.t. D).

Example: If $\lambda_1(L_j) \to 0$ or

$\operatorname{covol}(L_j) \to \infty$ or

$\operatorname{covol}(L_j) \to 0$

Then $L_j \to \infty$.

Prop $L_j \to \infty$ (in $\bar{\mathcal{X}}_n$) if and only if one of $\lambda_1(L_j) \to 0$, $\operatorname{covol}(L_j) \to \infty$, $\operatorname{covol}(L_j) \to 0$.

($\lambda_1(L_j) \to \infty$)

Pf We saw $\Uparrow$.

$\Downarrow$: Suppose by $\lambda_1(L_j) \not\to 0$, $\operatorname{covol}(L_j) \not\to \infty$

$\operatorname{covol}(L_j) \not\to 0$.

Passing to a subsequence, $\lambda_1(L_j)$ is bounded below, and $\operatorname{covol}(L_j)$ bounded above and below. Want to $L_j \to L$

(possibly passing to a further subsequence).

$\kappa_i(L)$ = length of i^{th} vector in Korkine-Zolotarev reduction procedure.

(ex. 7) $\kappa_i \asymp \lambda_i$

where $A \asymp B$ means A, B are functions on $\mathcal{X}_n$ and there is a constant $C \geq 1$ s.t. $\forall L \in \mathcal{X}_n$,

$$\frac{1}{C} A(L) \leq B(L) \leq C\, A(L).$$

(C may depend on n).

By Minkowski's second theorem

$$\lambda_1(L_j) \cdots \lambda_n(L_j) \asymp \operatorname{covol}(L_j)$$

bounded above and below ind. of j.

$$0 < \lambda_1(L_j) \leq \cdots \leq \lambda_n(L_j)$$

ind. of j.

$$\lambda_n(L_j) \leq C\, \frac{\operatorname{covol}(L_j)}{\lambda_1(L_j) \cdots \lambda_{n-1}(L_j)} \leq C\, \frac{\operatorname{covol}(L_j)}{\lambda_1(L_j)^{n-1}}$$

bounded above

$\Rightarrow \forall i \ \lambda_i(L_j)$ bounded above

$\Rightarrow \forall i \quad \lambda_i(L_j)$ bounded above.

If $\|v_i^{(j)}\| = \lambda_i(L_j)$, $L_j = \operatorname{span}_{\mathbb{Z}}(v_1^{(j)}, \dots, v_n^{(j)})$

Then (passing to a subsequence)

$$v_i^{(j)} \longrightarrow v_i$$

$\Rightarrow$ L_j is a convergent subseq. by (c) of the prop.

<u>Cor</u> (Mahler's compactness criterion a.k.a. Mahler's selection principle).

Let $\pi : SL_n(\mathbb{R}) \longrightarrow \mathcal{X}_n = SL_n(\mathbb{R})/SL_n(\mathbb{Z})$

$$g \longmapsto g\mathbb{Z}^n$$

Then for $S \subset SL_n(\mathbb{R})$, $\overline{\pi(S)}$ is compact

$\Leftrightarrow \exists\, \varepsilon > 0 \ \forall x \in S, \ \lambda_1(\pi(x)) \geq \varepsilon.$

<u>Equivalently</u>: Let $K_\varepsilon \subset \mathcal{X}_n$

$$K_\varepsilon = \{ L \in \mathcal{X}_n : \lambda_1(L) \geq \varepsilon \}$$

Then K_ε is compact for all $\varepsilon > 0$

and any $K \subset \mathcal{X}_n$ compact is contained

in K_ε for some $\varepsilon > 0$.

I.e. $\{ K_\varepsilon : \varepsilon > 0 \}$ are an exhaustion of $\mathcal{X}_n$.

Application

Cor $\alpha_r(L) = \inf \left\{ \operatorname{covol}(L_0) : \begin{array}{l} L_0 \subset L \text{ primitive} \\ \operatorname{rank}(L_0) = r \end{array} \right\}$

Then α_r is actually a minimum, i.e.

$\exists\, L_0 \subset L$ primitive of rank r, s.t.

$\operatorname{covol}(L_0) = \alpha_r(L)$.

Pf: Let $L_j \subset L$ be primitive subgroups

of rank r s.t. $\operatorname{covol}(L_j) \longrightarrow \alpha_r(L)$.

By ex. 7 $\exists$ a basis $u_1^{(j)}, \dots, u_r^{(j)}$ of

L_j s.t. $\| u_i^{(j)} \| = \lambda_i(L_j) \asymp \lambda_i(L_j)$

$\lambda_1(L) \le \lambda_1(L_j) \le \cdots \le \lambda_r(L_j) \le C \dfrac{\mathrm{covol}(L_j)}{\lambda_1(L_j)^{r-1}}$ for some C

using Mink. 2nd thm, as in previous pf.

So passing to subsequences, $u_i^{(j)} \longrightarrow u_i$ for all i

$u_i \in L$ because $u_i^{(j)} \in L$, L is closed.

Define $L_\infty = \mathrm{span}_{\mathbb{Z}}(u_1, \dots, u_r)$

$$\mathrm{covol}(L_\infty) = \| u_1 \wedge \cdots \wedge u_r \| = \lim_{j \to \infty} \| u_1^{(j)} \wedge \cdots \wedge u_r^{(j)} \|$$

$$= \lim_{j \to \infty} \mathrm{covol}(L_j) \longrightarrow \alpha_r(L)$$

So min is attained.

<u>Alternative proof</u>: $L = g\mathbb{Z}^n$

$L_i = \mathrm{span}_{\mathbb{Z}}(u_1^{(j)}, \dots, u_r^{(j)})$

$u_i^{(j)} = g v_i^{(j)}$ $\qquad v_i^{(j)} \in \mathbb{Z}^n$.

$v_1^{(j)} \wedge \cdots \wedge v_r^{(j)}$ is an element of $\Lambda^r \mathbb{R}^n = \mathbb{R}^n_r$

(the space of r-vectors in the Grassman algebre)

with integer coefficients with respect to the standard basis $\{e_\sigma : \sigma = (1 \le i_1 < \cdots < i_r \le n)\}$

Since $\|u_1^{(j)} \wedge \cdots \wedge u_r^{(j)}\|$ converges,

so does $\|v_1^{(j)} \wedge \cdots \wedge v_r^{(j)}\|$.

Since $\mathbb{Z}$-span of $\{e_\sigma\}$ is discrete,

$\|v_1^{(j)} \wedge \cdots \wedge v_r^{(j)}\|$ is eventually constant.

So $\|u_1^{(j)} \wedge \cdots \wedge u_r^{(j)}\|$ is eventually constant.

<u>Reminder</u> $\delta(L)$ = packing density of L

$$= \frac{\mathrm{Vol}\left(B\left(0, \frac{\lambda_1(L)}{2}\right)\right)}{\mathrm{covol}(L)} = \frac{V_n}{2^n} \frac{\lambda_1(L)^n}{\mathrm{covol}(L)}$$

= max. proportion of space filled by non-overlapping balls centred at pts of L.

$$\delta_n = \sup\{\delta(L) : L \in \mathcal{X}_n\},$$

Cor: sup is a max, i.e. in each dimension n there is L s.t. $\delta(L) = \delta_n$.

Pf: Looking for L_0 s.t.

$$\frac{\lambda_1(L_0)}{\mathrm{covol}(L_0)^{\frac{1}{n}}} \geq \frac{\lambda_1(L)}{\mathrm{covol}(L)^{\frac{1}{n}}} \quad \text{for all } L \in \mathcal{X}_n.$$

equivalently, $\mu(L_0) \overset{\text{def}}{=} \dfrac{\lambda_1(L_0)^2}{\mathrm{covol}(L_0)^{\frac{2}{n}}}$ is maximal.

this is the Hermite constant.

Suppose $L_j \in \mathcal{X}_n$ s.t. $L_j \xrightarrow[j\to\infty]{} \sup\{\lambda_1(L) : L \in \mathcal{X}_n\}$

So $\lambda_1(L_j) \geq c > 0$. By Mahler comp. criterion,

$L_j \to L_\infty$ along a subsequence

$$\Longrightarrow \lambda_1(L_j) \xrightarrow[j\to\infty]{} \lambda_1(L_\infty) = \max\{\lambda_1(L) ; L \in \mathcal{X}_n\}$$

(λ_1 is cts).

History of lattice packing problem

Hilbert's 18th problem, part 3

What is the densest sphere packing in $\mathbb{R}^3$?

What about other shapes?

Solved by Hales 1998.

Def: L_0 is called critical if

$$L \longmapsto \frac{\lambda_1(L)}{\operatorname{covol}(L)^{\frac{1}{n}}} \quad \text{achieves a local max}$$

at L_0. Extreme if L_0 achieves the global max (extreme $\Rightarrow$ critical).

Strategy followed for finding optimal lattices in dimensions $n = 2, \dots, 8$ is roughly the following:

(KZ)

<u>Step 1</u> If L_0 is critical then L_0 is <u>perfect</u> (will be defined later).

<u>Step 2</u> # of perfect lattices in any fixed dimension n is finite (up to dilations and orthogonal transformations). (Voronoi)

<u>Step 3</u> Enumerate all perfect lattices and find the one for which $\dfrac{\lambda_1(L)}{\mathrm{covol}(L)^{\frac{1}{n}}}$ is maximal.

(KZ ~ $n=4,5$ Blichfeldts $n=6,7,8$)

<u>Step 4</u> Show L is critical

$\Longleftrightarrow$ L is perfect and <u>eutactic</u> (will be defined later)

(Voronoi)

# of eutactic lattices in any dim is finite (again up to dilations and orth. trans.).

(Voronoi)

Symmetric matrices

$$\mathrm{Sym}_n = \{A \in M_n(\mathbb{R}) : A^t = A\}.$$

$= \{ A \in M_n(\mathbb{R}) : \forall x, y \in \mathbb{R}^n, \langle Ax, y\rangle = \langle x, Ay\rangle \}$

Sym_n is a real vector space of dim $\frac{n(n+1)}{2}$.

Facts: • Any $A \in Sym_n$ can be diagonalized over $\mathbb{R}$, by an orthonormal transformation.

i.e. $\exists O \in O_n(\mathbb{R})$ s.t. $O^t A O$ is diagonal.

orthonormal matrices, $O_n(\mathbb{R}) = \{ g \in GL_n(\mathbb{R}) : g^t g = I_d \}$

• $A \in Sym_n$ called positive definite if the eigenvalues are positive

$\Longleftrightarrow \forall x \in \mathbb{R}^n, \langle Ax, x\rangle \geq 0, \langle Ax, x\rangle = 0 \Leftrightarrow x = 0$

$\Longleftrightarrow \exists B \in GL_n(\mathbb{R})$ s.t. $A = B^t B$.

Pf of last equivalence.

$\Uparrow$ $A = B^t B$, then $\forall x \in \mathbb{R}^n$ $\qquad \forall x \in \mathbb{R}^n - \{0\}$

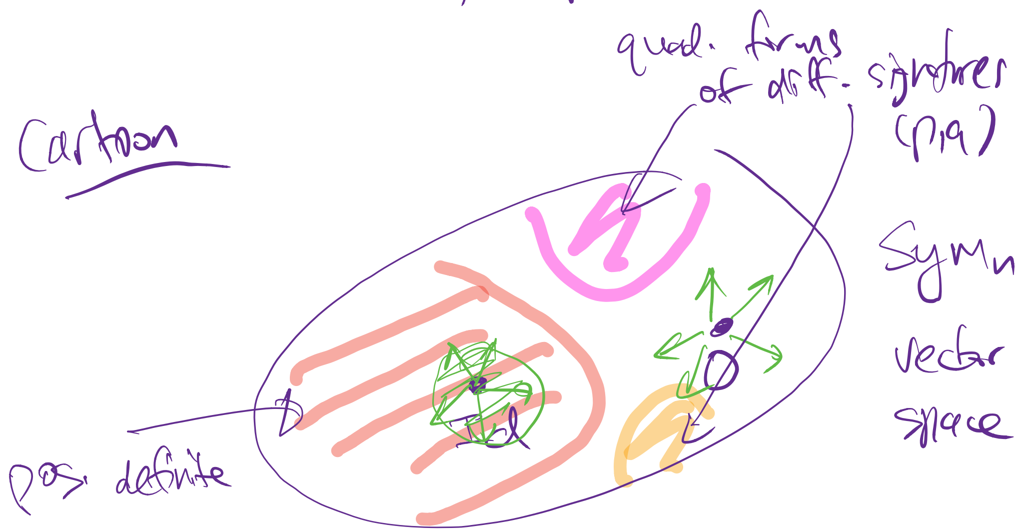
$\langle Ax, x\rangle = \langle x, Ax\rangle = \langle x, B^t B x\rangle = \langle Bx, Bx\rangle > 0$

$\Downarrow$ If $A = O^t D O$, $D = \mathrm{diag}(\beta_1, \ldots, \beta_n)$ $\beta_i > 0$

Define $\sqrt{D} = \mathrm{diag}(\sqrt{\beta_1}, \ldots, \sqrt{\beta_n})$

Define $B = O^t \sqrt{D}\, O$, compute $B^t B = A$.

Cartoon



For us, Sym_n will play the role of

tangent space at Id of pos. def. matrices.

For $x \in \mathbb{R}^n$, define $\varphi_x : \mathrm{Sym}_n \longrightarrow \mathbb{R}$

$$\varphi_x(A) = \langle Ax, x \rangle$$

$$\varphi_x \in (\mathrm{Sym}_n)^*$$

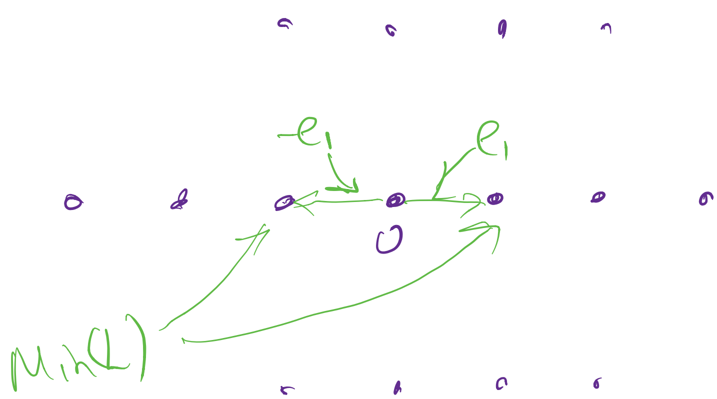
Def Let $F \subset \mathbb{R}^n$. F is called *perfect* if

$$(\mathrm{Sym}_n)^* = \mathrm{span}\left(\{\varphi_x : x \in F\}\right).$$

The vectors $\{v \in L : \lambda_1(L) = \|v\|\}$ are the *minimizers for L*, notation: $\mathrm{Min}(L)$.

L is *perfect* if $\mathrm{Min}(L)$ is perfect.

Examples: (a) $L = \mathbb{Z}\binom{1}{0} \oplus \mathbb{Z}\binom{0}{2}$.



$$\mathrm{Sym}_2 = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$

$$\varphi_{e_1}(A) = \left\langle \binom{1}{0}, \begin{pmatrix} a & b \\ b & c \end{pmatrix} \binom{1}{0} \right\rangle = \left\langle \binom{1}{0}, \binom{a}{b} \right\rangle = a$$

$(a,b,c) \longmapsto a.$

not perfect.

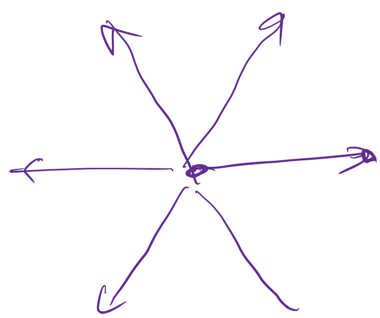
(b) $L = \mathbb{Z}^2 \quad \mathrm{Min}(L) = \{\pm e_1, \pm e_2\}.$

$\varphi_{e_2}(A) = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = c.$

not perfect.

(two functionals can't span a 3-dim space).

(c)



L = hex. lattices

$|\mathrm{Min}(L)| = 6.$

ex. L is perfect.

<u>Prop</u>: If $F \subset \mathbb{R}^n$ is perfect then $\mathbb{R}^n = \mathrm{span}(F)$.

(not iff as example (b) shows).

Pf: Suppose by contradiction $V = \mathrm{span}(F) \subsetneq \mathbb{R}^n$

Let $A \neq 0$ $V \subset \ker A$. Can take A

symmetric (ex.).

Then $\forall x \in F$, $\varphi_x(A) = \langle Ax, x \rangle = \langle 0, x \rangle = 0$

Contradiction to $\{\varphi_x : x \in F\}$ spans $(\mathrm{Sym}_n)^*$.

Prop: If L is perfect then

$|\mathrm{Min}(L)| \geq n(n+1)$.

Pf: Let $\mathrm{Min}(L) = \{\pm v_1, \dots, \pm v_t\}$

$|\mathrm{Min}(L)| = 2t$. $\varphi_{v_i} = \varphi_{-v_i}$ $\forall i$

$\mathrm{span}(\{\varphi_x : x \in \mathrm{Min}(L)\}) = \mathrm{span}(\{\varphi_{v_i} : i = 1, \dots, t\})$

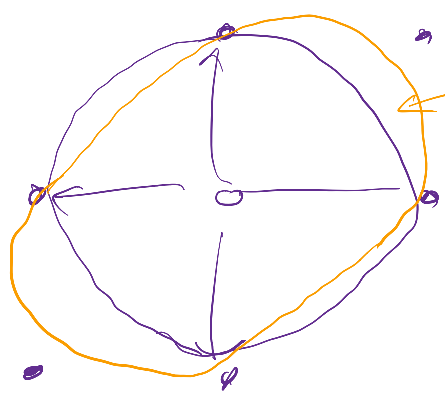
has dim $\frac{n(n+1)}{2}$ $\Longrightarrow t \geq \frac{n(n+1)}{2} \Longrightarrow |\mathrm{Min}(L)| = 2t$
$\geq n(n+1)$.

From this, easy to show (ex.) that hexagonal lattice is the unique (up to dilation and rotation) perfect lattice in $\mathbb{R}^2$.

<u>Thm</u> (Korkine–Zolotarev)

If L is critical then L is perfect.

<u>Idea</u> (picture for $\mathbb{Z}^2$ lattice)



Apply a lin. trans. g that maps E to

$B(0,r)$ $\quad r>1$, $\det g = 1$.

$g\mathbb{Z}^2$ is close to $\mathbb{Z}^2$ because E is close to $B(0,1)$, and $\lambda_1(g\mathbb{Z}^2) > 1$.

$g \in GL_n(\mathbb{R})$

Ellipsoid $g(B(0,1)) =$

$= g\{x \in \mathbb{R}^n : \langle x, x\rangle \leq 1\}$

$= \{y \in \mathbb{R}^n : \langle g^{-1}x, g^{-1}x\rangle \leq 1\} = \{y \in \mathbb{R}^n : \langle Ay, y\rangle \leq 1\}$

$A = (g^{-1})^t g^{-1}$.

<u>Prop</u>: Define $GL_n(\mathbb{R}) \longrightarrow Sym_n$

$g \longmapsto A(g) = g^t g - Id$.

There is a nbd $\mathcal{U}$ of Id in $GL_n(\mathbb{R})$ s.t. $\mathcal{V} = \{A(g) : g \in \mathcal{U}\}$ is a nbd of 0 in Sym_n, and $g \longmapsto A(g)$ restricted to $\mathcal{U}$ is an open surjective smooth $\mathcal{U} \to \mathcal{V}$.

$(g \in \mathcal{U},\ A(g) = 0) \iff g \in O_n$.

orthogonal matrices.

Pf: Open mapping theorem. (ex.)

($\circledast$ is an inverse of $g \mapsto A(g)$,

as long as $\mathrm{Id}+A$ is pos. definite).

Lemma 1 For any lattice L, there is a nbd $\mathcal{U}$ of Id in $GL_n(\mathbb{R})$ s.t.

$$\forall g \in \mathcal{U}, \quad \mathrm{Min}(gL) \subset g\,\mathrm{Min}(L).$$

Pf: Since L is discrete, there is $\eta > 0$ s.t. if $v \in L$, $v \neq 0$, $v \notin \mathrm{Min}(L)$ then $\|v\| > (1+\eta)\lambda_1(L)$.

Define $\mathcal{U} = \left\{ g \in GL_n(\mathbb{R}) : \|g - \mathrm{Id}\|_{op} < \frac{\eta}{2(1+\eta)} \right\}$

Then $\forall v \in \mathrm{Min}(L)$, $\forall g \in \mathcal{U}$, $\|v\| = \lambda_1(L)$

$$\|gv\| \le \|v\| + \|(g-\mathrm{Id})v\| < \left(1+\frac{\eta}{2(1+\eta)}\right)\|v\|$$
$$< \left(1+\frac{\eta}{2}\right)\lambda_1(L).$$

$\forall v \in L,\ v \neq 0,\ v \notin \mathrm{Min}(L)\ \forall g \in \mathcal{U},$

$$\|gv\| \geq \|v\| - \|(g - \mathrm{Id})v\| \geq \|v\| - \frac{\eta}{2(1+\eta)}\|v\|$$

$$= \|v\|\left[\frac{2+\eta}{2(1+\eta)}\right] \geq \lambda_1(L)(1+\eta)\left(\frac{1+\eta/2}{1+\eta}\right) = \left(1+\frac{\eta}{2}\right)\lambda_1(L)$$

So the shortest vector of $gL \setminus \{0\}$ is of the form gv for some $v \in \mathrm{Min}(L)$.

<u>Lemma 2</u> For any L, there is a nbd $\mathcal{V}$ of 0 in Sym_n s.t. for $A = A(g) \in \mathcal{V}$

$$\lambda_1(gL)^2 = \lambda_1(L)^2 + \min_{v \in \mathrm{Min}(L)} \varphi_v(A).$$

<u>Pf</u>: By Lemma 1, in order to compute $\lambda_1(gL)$ (where $g \in \mathcal{U}$), suffices to consider vectors gv, $v \in \mathrm{Min}(L)$.

For such v,

$$\|gv\|^2 = \langle gv, gv\rangle = \langle v, g^t g v\rangle = \langle v, (\mathrm{Id}+A)v\rangle$$

$$= \|v\|^2 + \langle v, Av \rangle = \|v\|^2 + \varphi_v(A).$$

We are trying to maximize $\dfrac{\lambda_1(L)}{\mathrm{covol}(L)^{\frac{1}{n}}}$.

To understand denominator, need to understand $\det(g)$ for g close to Id.

<u>Lemma 3</u>: There is a nbd $\mathcal{V}$ of 0 in Sym_n such that for any $A = A(g) \in \mathcal{V}$, with $\mathrm{Tr}(A) \leq 0$, either $g \in O_n$, or $\det g < 1$.

<u>Rk</u> If $A(g_1) = A(g_2)$ then $g_1 = O g_2$ where $O \in O_n$ and $|\det(g_1)| = |\det(g_2)|$.

Let's postpone the proof.

Proof of KZ thm (assuming lemma 3).

Suppose L is critical, and suppose (by contradiction) that $\exists A \in \mathrm{Sym}_n$, $A \neq 0$ s.t. $\forall x \in \mathrm{Min}(L)$, $\varphi_x(A) = 0$.

By replacing (if necessary) A with $-A$, we can $\mathrm{tr}(A) \leq 0$. By replacing A with sA for $s > 0$ small, can find such A arbitrarily close to 0.

For each small s, let $g_s \in GL_n(\mathbb{R})$ s.t. $A(g_s) = sA$. Since $sA \neq 0$, $g_s \notin O_n$. By Lemma 3, $\det(g_s) < 1$.

By Lemma 2,

$\lambda_1(g_s L) = \lambda_1(L)$.

So $$\frac{\lambda_1(g_s L)}{\operatorname{covol}(g_s L)^{\frac{1}{n}}} = \frac{\lambda_1(L)}{\operatorname{covol}(L)^{\frac{1}{n}}} \cdot \frac{1}{\det(g_s)^{\frac{1}{n}}}$$

$$> \frac{\lambda_1(L)}{\operatorname{covol}(L)^{\frac{1}{n}}}$$

$\therefore$ L is not critical.