Geometry of Numbers, Lecture 8

What is the optimal lattice packing by Euclidean balls in $\mathbb{R}^n$?

$$\delta_n = \max\left\{ \operatorname{Vol}(B) : \begin{array}{l} B = B(0,r) \text{ a Euclidean ball} \\ \exists L \in \mathcal{X}_n \text{ s.t. } \{B+\ell : \ell \in L\} \\ \text{are disjoint} \end{array} \right\}.$$

Known only in dimensions $n \in \{1, \dots, 8\} \cup \{24\}$.

Equivalently, want to understand:

$$\max\{\lambda_1(L) : L \in \mathcal{X}_n\}$$

or $\max\{\lambda_1(L)^2 : L \in \mathcal{X}_n\}$ ← Hermite constant in dim. n.

<u>Notation</u> $r_{\text{eff}} = r_{\text{eff}}^{(n)}$ is the value of r for which $\operatorname{Vol}(B(0,r)) = 1$.

If V_n is the volume of $B(0,1)$ in $\mathbb{R}^n$ then $V_n r^n = 1 \iff r_{\text{eff}} = V_n^{-\frac{1}{n}} \sim \frac{\sqrt{n}}{\sqrt{2\pi e}}$

<u>Thm</u> (Minkowski–Hlawka): For any n $\exists L \in \mathcal{X}_n$ s.t. $\lambda_1(L) \geq r_{\text{eff}}^{(n)}$.

Equivalently: $\delta_n \geq \frac{1}{2^n}$

Because: $\lambda_1(L) \geq r_{\text{eff}}^{(n)} \Rightarrow \{\ell + B : \ell \in L\}$ where $B = B(0, \frac{r_{\text{eff}}^{(n)}}{2})$ are disjoint.

$$\text{Vol}\left(B\left(0, \frac{r_{\text{eff}}}{2}\right)\right) = \frac{1}{2^n} \text{Vol}(B(0, r_{\text{eff}})) = \frac{1}{2^n}.$$

Remarks: ① Currently best known bounds on δ_n are:

$$C\frac{n}{2^n} \leq \delta_n \leq (0.67)^n$$

Kabatyanski and Levenshtein '70s.

many contributions.
Best value of C due to Venkatesh 2012.

② History: Minkowski 1911 stated without proof.
Proved by Hlawka '43.

Reproved by Siegel in '45 (apparently filling in details of Minkowski's argument).

③ "Probabilistic argument". Proves existence.
No explicit lattices with $\lambda_1(L) \geq r_{\text{eff}}$ are known.

Thm A (Siegel summation formula '45)

There is a measure $m_{\mathcal{X}_n}$ on $\mathcal{X}_n$

which is : (i) $SL_n(\mathbb{R})$-invariant, regular, Borel.

(ii) a probability measure.

(iii)

$\forall f \in L^1(\mathbb{R}^n, \mathrm{Vol})$, define $\hat{f} : \mathcal{X}_n \longrightarrow \mathbb{R}$

$$\hat{f}(L) = \sum_{v \in L \setminus \{0\}} f(v).$$ $\hat{f}$ is well-defined

and $$\int_{\mathcal{X}_n} \hat{f} \, dm_{\mathcal{X}_n} = \int_{\mathbb{R}^n} f \, d\mathrm{Vol}. \quad \text{(SSF)}$$

Pf of Minkowski-Hlawka, assuming Theorem A

Suppose M-H is false, i.e. for every $L \in \mathcal{X}_n$, $\lambda_1(L) < r_{\text{eff}}$. This means $\forall L \in \mathcal{X}_n$ $L \cap B(0, r_{\text{eff}}) \neq \{0\}$. So $L \cap B(0, r_{\text{eff}})$ contains at least two nonzero vectors $\pm v$.

Let $f = \mathbb{1}_{B(0, r_{eff})}$, by (SSF)

$$1 = \mathrm{Vol}(B(0, r_{eff})) = \int_{\mathbb{R}^n} f \, d\mathrm{Vol} = \int_{\mathcal{X}_n} \hat{f} \, dm_{\mathcal{X}_n} \geq 2$$

$$\hat{f}(L) = \sum_{v \in L \setminus \{0\}} \mathbb{1}_{B(0, r_{eff})}(v) \geq 2$$

contradiction.

<u>Remark:</u> * Can improve const. very slightly by replacing 1 on the LHS with $2 - \delta$.

* If $A \subset \mathbb{R}^n$ measurable, $\mathrm{Vol}(A) < 1$ then $\exists\, L \in \mathcal{X}_n$ s.t. $(L \setminus \{0\}) \cap A = \emptyset$.

<u>"Reminder" about measure theory.</u>

$(X, \mathcal{B})$ X a set $\mathcal{B}$ a σ-algebra

$\mathcal{B} \subset \mathcal{P}(X)$ closed under countable unions and intersections, complements, contains $\emptyset$, X.

a measure μ on $(X, \mathcal{B})$ is a function

$\mu : \mathcal{B} \to [0, \infty]$

$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$. $\left(\Rightarrow (A \subset B \Rightarrow \mu(A) \leq \mu(B))\right)$.

In our setup, X will be locally compact second countable (lcsc). A special case of lcsc is: X is a metric space in which closed balls $\bar{B}(x,r) = \{y \in X : d(x,y) \leq r\}$ are compact. Main example: $\mathcal{X}_n$ with Chabauty-Fell metric.

$\mathcal{B}$ will: Borel σ-algebra - smallest σ-algebra containing the open sets. Then μ is called Borel.

$(X, \mathcal{B}, \mu)$: μ is locally finite if

lcsc space, Borel, measure

$\forall x \in X$ $\exists$ nbd U of x s.t. $\mu(U) < \infty$.

In lcsc spaces $\Longleftrightarrow$ $\mu(K) < \infty$ for any compact set K.

μ is called <u>inner regular</u> if for any U open set, $\mu(U) = \sup\{\mu(K) : K \subset U \text{ compact}\}$

μ is called <u>outer regular</u> if $\forall S \in \mathcal{B}$,

$$\mu(S) = \inf\{\mu(U) : S \subset U \text{ open}\}$$

<u>Def</u> μ is called <u>Radon</u> if it is Borel, locally finite, inner and outer regular

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Let $C_c(X) = \left\{ \begin{array}{l} \text{continuous compactly} \\ \text{supported functions } X \to \mathbb{R} \end{array} \right\}$

Given a Radon measure μ on X,

define $I_\mu : C_c(X) \longrightarrow \mathbb{R}$

$$I_\mu(f) = \int_X f \, d\mu$$

I_μ is a continuous positive linear functional on $C_c(X)$.

Thm (Riesz representation thm, a.k.a. Riesz-Kakutani-Markov rep'n thm)

The mapping

$$\left\{\begin{array}{l}\text{Radon measures}\\ \text{on } X\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{continuous positive}\\ \text{linear functionals on}\\ C_c(X)\end{array}\right\}$$

$$\mu \longmapsto I_\mu$$

is a bijection.

Let $\varphi : C_c(X) \to \mathbb{R}$ be a linear functional.

φ is positive if $f \geq 0 \Rightarrow \varphi(f) \geq 0$.

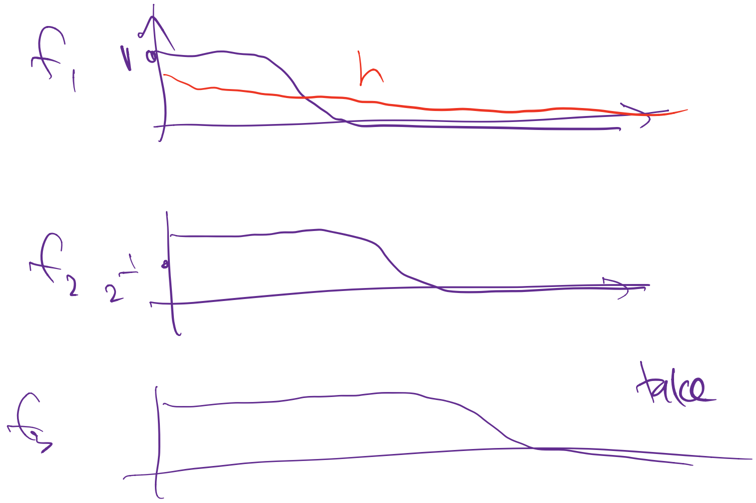
φ is continuous if $\forall K \subset X$ compact $\exists C > 0$ s.t. if f vanishes outside of K then

$$|\varphi(f)| \leq C\|f\|_\infty .$$

Clearly I_μ is continuous, for each K take $C = \mu(K)$.

<u>Remark</u>: If X is compact then continuity just means φ is a bounded linear functional on $C(X) \leftarrow$ Banach space with sup-norm.

Note $C_c(X)$ with sup-norm is not complete when X is not compact. For example $X = \mathbb{R}_{\geq 0}$.



f_i take values in $[0,1]$ and supported on $[0,i]$

take $h(x) > 0$, $h(x) \xrightarrow[x\to\infty]{} 0$

Then $h \cdot f_i \xrightarrow[i\to\infty]{} h$ in sup-norm

but $h \notin C_c(\mathbb{R}_{\geq 0})$

$$C_c(X) = \bigcup_{\substack{K \subset X \\ \text{compact}}} \underbrace{\{f : X \to \mathbb{R} \text{ cts.}, \operatorname{supp} f \subset K\}}_{\text{can be equipped with sup-norm}}$$

$C_c(X)$ is a direct limit of Banach spaces

With the direct limit topology it becomes a LCTVS.

Cor In order to define a Radon measure on X, suffices to define a continuous pos lin. functional on $C_c(X)$.

Prop: $C_c(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n, \mathrm{Vol})$.

Cor: For any Radon measure μ on $\mathfrak{X}_n$, if (SSF) holds for μ and for $f \in C_c(\mathbb{R}^n)$, then it holds for all $f \in L^1(\mathbb{R}^n)$, and $\hat{f} \in L^1(\mu)$.

Pf: $\varphi : C_c(\mathbb{R}^n) \longrightarrow L^1(\mathfrak{X}_n, \mu)$ $\varphi(f) = \hat{f}$

$\psi : C_c(\mathbb{R}^n) \longrightarrow \mathbb{R}$ $\psi(f) = \|\hat{f}\|_{L^1(\mathbb{R}^n, \mathrm{Vol})}$

$\hat{f}(L) = \sum_{v \in L \setminus 0} f(v)$ (note the sum in def of $\hat{f}$ converges because L is discrete and f is compactly supported, so sum is finite)

$\hat f$ is continuous as a fn of L.

(if B is a large ball properly containing supp f, and $L_j \xrightarrow[j\to\infty]{} L$ (in CF metric)

then for all large enough j, $\forall x \in B\cap L$

$\exists\, x_j \in L_j \cap B$, $x_j \to x$, and $B \cap L_j = \{x_j\}$.

$f(x_j) \to f(x)$

and so the sum in def. of $\hat f(L)$,

$\hat f(L_j)$ is finite with a bound on the number of nonzero summands).

We are assuming

$$\forall f \in C_c(\mathbb{R}^n) \quad \int_{\mathcal{X}_n} \hat f \, d\mu = \int_{\mathbb{R}^n} f \, d\mathrm{Vol} \quad \forall f \in C_c(\mathbb{R}^n)$$

$$\|\varphi(f)\|_{L^1(\mu)} = \int_{\mathcal{X}_n} |(\hat f)| \, d\mu \le \int_{\mathcal{X}_n} (|f|)^{\wedge} \, d\mu \overset{(SSF)}{=} \int_{\mathbb{R}^n} |f| \, d\mu = \|f\|_{L^1(\mathrm{Vol})}$$

Similarly (check!) $|\varphi(f)| \le \|f\|_{L^1(\mathrm{Vol})}$

φ, ψ are linear functionals on $C_c(\mathbb{R}^n)$ which are L^1 bounded. Since $C_c(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n, Vol)$, there is a unique continuous extension of φ, ψ to all $f \in L^1(\mathbb{R}^n, Vol)$.

It can be checked (ex.) that the extension of φ, ψ to $L^1(\mathbb{R}^n, Vol)$ is given by the same formula.

Radon-Nikodym thm

Let μ_1, μ_2 be Radon measures on lcsc X. We say μ_1 is abs. continuous w.r.t. μ_2 (notation $\mu_1 \ll \mu_2$) if $\forall A \in \mathcal{B}$, $\mu_2(A) = 0$

$\Longrightarrow \mu_1(A) = 0$.

Thm (Radon-Nikodym) If $\mu_1 \ll \mu_2$ as above then $\exists$ measurable $\varphi : X \to \mathbb{R}_{\geq 0}$ s.t. for $\forall A \in \mathcal{B}$, $\mu_1(A) = \int_A \varphi \, d\mu_2$ $(*)$

φ is unique up to ~~zero~~ measure, i.e. if φ, φ' satisfy $(*)$ then $\varphi(x) = \varphi'(x)$ for μ_2-a.e. x.

Notation: $\varphi(x) = \frac{d\mu_1}{d\mu_2}(x)$.

<u>Example</u>: When $X = \mathbb{R}^N$, $\mu_1 = \mathrm{Vol}$, and $g \in GL_n(\mathbb{R})$, $\mu_2(A) = \mu_1(g^{-1}(A))$, then

$$\frac{d\mu_1}{d\mu_2} = |\det(g)|. \quad (= \text{Jacobian formula}).$$

Recall: G a group, X a set.

An action of G on X (<u>notation</u> $G \curvearrowright X$) is a map $G \times X \longrightarrow X$ (action map)

$$(g, x) \longmapsto gx$$

s.t. $g_1(g_2 x) = (g_1 g_2)x$ and $e_G x = x \; \forall x \in X$
$\forall g_1, g_2 \in G \; x \in X$

In our setup, G and X will be lcsc, and action map is required to be continuous.

<u>Example:</u> $G = GL_n(\mathbb{R})$, $d(g_1, g_2) = \|g_1 - g_2\|_{op}$

Any closed subgroup is lcsc.

If G acts on X, it also acts on Radon measures on X, by $g_* \mu(A) = \mu(g^{-1}(A))$ where $g \in G$, $A \in \mathcal{B}$, μ is Radon.

$g_* \mu$ is called the pushforward of μ by g.

In terms of R.R.T., $\int_X f \, dg_* \mu =$

$$= \int_X f \circ g \, d\mu \qquad [f \circ g(x) = f(gx)].$$

The identity $\int_X f \circ g \, d\mu = \int_X f \, dg_* \mu$ holds for all $f \in L^1(g_* \mu)$.

<u>Def</u> μ is G-invariant if $g_* \mu = \mu$ for all $g \in G$.

$G \curvearrowright X$ is called <u>transitive</u> if $\forall x_1, x_2 \in X$
$\exists g \in G$ s.t. $g x_1 = x_2$.

<u>Prop</u>: If $G \curvearrowright X$ transitive, then any two nonzero G-inv. Radon measures are proportional, i.e. if μ_1, μ_2 are both G-inv. and Radon, and nonzero, then $\exists c > 0$ s.t. $\forall A \in \mathcal{B}$, $c\mu_1(A) = \mu_2(A)$.

<u>Proof</u> (sketch, details in ex.):

Assume $\mu_1 \ll \mu_2$. Let $\varphi = \frac{d\mu_1}{d\mu_2}$.

Using the G-invariance of μ_1, μ_2 can show that φ is constant a.e.

More precisely, after modifying φ on a subset of measure 0 w.r.t. μ_2, φ is G-inv. (exercise).

By transitivity, φ is constant.

$\Rightarrow c_1\mu_1 = \mu_2$ where c is the constant value of φ.

If μ_1 is not abs. cts. w.r.t. μ_2, define $\mu_3 = \mu_1 + \mu_2$. Now $\mu_i \ll \mu_3$ $i=1,2$

By previous case, $\exists$ constants c_1, c_2 s.t.

$\mu_3 = c_1\mu_1 = c_2\mu_2 \Rightarrow \mu_1, \mu_2$ are proportional.

Examples (0) $X = \mathbb{R}^n$, $G = \mathbb{R}^n$

G acts by translations.

Vol is the unique (up to scaling) G-inv. Borel measure on $\mathbb{R}^n$.

(1) $X = \mathbb{R}^n$, $G = SL_n(\mathbb{R})$, action by linear transformations.

$G \curvearrowright \mathbb{R}^n \setminus \{0\}$. $G \curvearrowright \{0\}$.

On $\mathbb{R}^n \setminus \{0\}$ can use Vol.

By Jacobian formula, G preserves Vol on $\mathbb{R}^n \setminus \{0\}$.

On $\{0\}$, δ_0 (Dirac measure on $\{0\}$) is G-inv.

Therefore, any G-inv. Radon measure on $\mathbb{R}^n$ is of the form $c_1 \mathrm{Vol} + c_2 \delta_0$.

(3) $G = SL_n(\mathbb{R})$ (or any other group) acts on itself by left and right translations,

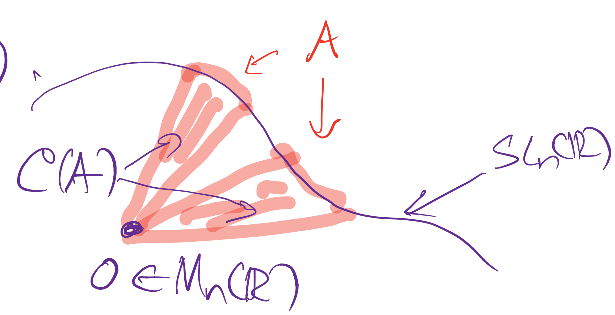
$$L_g(g_0) = g g_0, \quad R_g(g_0) = g_0 g^{-1}$$

$$L_{g_1} \circ L_{g_2} = L_{g_1 g_2} \quad R_{g_1} \circ R_{g_2} = R_{g_1 g_2}$$

Given $A \subset SL_n(\mathbb{R})$ (Borel)

define $C(A) = \{ta : a \in A,\ t \in [0,1]\}$

("the cone over A").



Define $m_{SL_n(\mathbb{R})}(A) = \mathrm{Vol}_{M_n(\mathbb{R})}(C(A))$

$(M_n(\mathbb{R}) \cong \mathbb{R}^{n^2})$

<u>Claim:</u> $m_{SL_n(\mathbb{R})}$ is inv. under both L_g and R_g

for any $g \in SL_n(\mathbb{R})$.

<u>Pf</u>: Note $\forall g \in G$, $C(L_g(A)) = gC(A)$

$$C(R_g(A)) = C(A)g^{-1}$$

So to prove claim, suffices to show

that for any $S \subset M_n(\mathbb{R})$ (Borel), $\forall g \in SL_n(\mathbb{R})$.

$$\mathrm{Vol}_{M_n(\mathbb{R})}(gS) = \mathrm{Vol}_{M_n(\mathbb{R})}(S) = \mathrm{Vol}_{M_n(\mathbb{R})}(Sg^{-1})$$

Indeed, write a matrix $\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$

$$= \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^{n^2}$$

In these coordinates L_g acts on $M_n(\mathbb{R})$ by $\begin{pmatrix} g & & & 0 \\ & g & & \\ & & \ddots & \\ 0 & & & g \end{pmatrix}$

The Jacobian of $L_g : M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$

is $(\det g)^n = 1$ because $g \in SL_n(\mathbb{R})$.

Similarly, R_g preserves $\mathrm{Vol}_{M_n(\mathbb{R})}$

and hence preserves $m_{SL_n(\mathbb{R})}$.

Remark: If G is lcsc it has a right Haar measure and a left measure
ν_R ν_L

They are the unique (up to proportionality) measures on G which are R_g and L_g inv., for any $g \in G$.

If $\nu_L = \nu_R$ (up to proportionality) then G is called unimodular. We exhibited a measure on $SL_n(\mathbb{R})$ which is left- and right-inv., hence is Haar measure, and $SL_n(\mathbb{R})$ is unimodular.

(4) $U = \{ g \in SL_n(\mathbb{R}) : g e_1 = e_1 \}$ $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

$$U = \left\{ \left(\begin{array}{c|c} 1 & x \\ \hline 0 & \\ \vdots & A \\ 0 & \end{array} \right) : A \in SL_{n-1}(\mathbb{R}) \quad x \in \mathbb{R}^{n-1} \right\}$$

$$m_U(S) = \int_{SL_{n-1}(\mathbb{R})} \int_{\mathbb{R}^{n-1}} \mathbb{1}_S \left(\begin{array}{c|c} 1 & x \\ \hline 0 & A \end{array} \right) d\mathrm{Vol}(x)\, dm_{SL_{n-1}(\mathbb{R})}(A)$$

m_U is both left and right invariant.

To see this, let's compute the group law on U: Notation $(A, x) \longleftrightarrow \left(\begin{array}{c|c} 1 & x \\ \hline 0 & A \end{array} \right)$

$$(A_1, x_1) \cdot (A_2, x_2) = \left(\begin{array}{c|c} 1 & x_1 \\ \hline 0 & A_1 \end{array} \right) \left(\begin{array}{c|c} 1 & x_2 \\ \hline 0 & A_2 \end{array} \right) = \left(\begin{array}{c|c} 1 & x_2 + x_1 A_2 \\ \hline 0 & A_1 A_2 \end{array} \right)$$

right mult. by A

$$= (A_1 A_2, x_2 + x_1 A_2)$$

(this is an example of a semi direct product).

$$\int_U f \circ L_{(A_0, x_0)}(A, x)\, dm_U(A, x) =$$

$$= \int_{SL_{n-1}(\mathbb{R})} \int_{\mathbb{R}^{n-1}} f(A_0 A, x + x_0 A)\, d\mathrm{Vol}_{\mathbb{R}^{n-1}}(x)\, dm_{SL_{n-1}(\mathbb{R})}(A)$$

$$= \int_{SL_{n-1}(\mathbb{R})} \int_{\mathbb{R}^{n-1}} f(A_0 A)\, dVol_{\mathbb{R}^{n-1}}(x)\, dm_{SL_{n-1}(\mathbb{R})}(A)$$

Vol is inv. by translation by $x_0 A$

$$= \int_{SL_{n-1}(\mathbb{R})} \int_{\mathbb{R}^{n-1}} f(A)\, dVol_{\mathbb{R}^{n-1}}(x)\, dm_{SL_{n-1}(\mathbb{R})}(A)$$

$m_{SL_{n-1}(\mathbb{R})}$ is L_{A_0}-inv.

$$= \int_{\mathcal{U}} f(A,x)\, dm_{\mathcal{U}}(A,x).$$

Similarly for right translation:

$$(A_0, x_0)^{-1} = (A_0^{-1}, -x_0 A_0^{-1})$$

$$\int_{\mathcal{U}} f \circ R_{(A_0,x_0)}(A,x)\, dm_{\mathcal{U}}(A,x) =$$

$$= \int_{SL_{n-1}(\mathbb{R})} \int_{\mathbb{R}^{n-1}} f(AA_0^{-1}, xA_0^{-1} - x_0 A_0^{-1})\, dVol_{\mathbb{R}^{n-1}}(x)\, dm_{SL_{n-1}(\mathbb{R})}(A)$$

$$= \int_{SL_{n-1}(\mathbb{R})} \int_{\mathbb{R}^{n-1}} f(AA_0^{-1}, xA_0^{-1})\, dVol_{\mathbb{R}^{n-1}}(x)\, dm_{SL_{n-1}(\mathbb{R})}(A)$$

$Vol_{\mathbb{R}^{n-1}}$ is inv under translation by $-x_0 A_0^{-1}$

$$= \int_{SL_{n-1}(\mathbb{R})} \int_{\mathbb{R}^{n-1}} f(AA_0^{-1}, x)\, dVol_{\mathbb{R}^{n-1}}(x)\, dm_{SL_{n-1}(\mathbb{R})}(A)$$

$\mathrm{Vol}_{\mathbb{R}^{n-1}}$ is inv. under right mult. by A_0^{-1} since $\det(A_0^{-1})=1$

$m_{SL_{n-1}(\mathbb{R})}$ is $R_{A_0^{-1}}$-inv

$$= \int_{SL_{n-1}(\mathbb{R})} \int_{\mathbb{R}^{n-1}} f(A,x)\, d\mathrm{Vol}_{\mathbb{R}^{n-1}}(x)\, d m_{SL_{n-1}(\mathbb{R})}(A)$$

$$= \int_{\mathcal{U}} f(A,x)\, dm_{\mathcal{U}}(A,x)$$

5 The affine group

$ASL_n(\mathbb{R}) =$

$$= \left\{ \left(\begin{array}{c|c} A & x \\ \hline 0 \cdots 0 & 1 \end{array} \right) : A \in SL_n(\mathbb{R}),\ x \in \mathbb{R}^n \right\}$$

<u>Group law:</u> $(A_1, x_1)(A_2, x_2) = \left(\begin{array}{c|c} A_1 & x_1 \\ \hline 0 & 1 \end{array} \right) \left(\begin{array}{c|c} A_2 & x_2 \\ \hline 0 & 1 \end{array} \right)$

$$= \left(\begin{array}{c|c} A_1 A_2 & x_1 + A_1 x_2 \\ \hline 0 & 1 \end{array} \right) = (A_1 A_2,\ A_1 x_2 + x_1)$$

Define $m_{ASL_n(\mathbb{R})}$ by setting (for $f \in C_c(ASL_n(\mathbb{R}))$)

$$\int_{ASL_n(\mathbb{R})} f(A,x)\, dm_{ASL_n(\mathbb{R})} = \int_{SL_n(\mathbb{R})} \int_{\mathbb{R}^n} f(A,x)\, d\mathrm{vol}_{\mathbb{R}^n}(x)\, dm_{SL_n(\mathbb{R})}(A).$$

A similar computation to the one in (4) shows $m_{ASL_n(\mathbb{R})}$ is both left- and right-invariant.