Geometry of Numbers, Lecture 8
What is the optimal lattice packing by Euclidean balls in $\mathbb{R}^n$?

$$\delta_n = \max \left\{ \text{Vol}(B) : B = B(x, r) \text{ a Euclidean ball in } \mathbb{R}^n \text{ st. } B + k \cdot L \text{ are disjoint} \right\}.$$

Known only in dimensions $n = 3, 4, 8, 24$.

Equivalently, want to understand:

$$\max \left\{ \Upsilon(L) : L \subseteq \mathbb{R}^n \right\}$$

or

$$\max \left\{ \Upsilon(L)^2 : L \subseteq \mathbb{R}^n \right\} \leq \text{Hermite constant in dim. } n.$$

\underline{Notation} \quad r_{\text{eff}} = r_{\text{eff}}^\text{con}\text{ is the value of } r \text{ for which } \text{Vol}(B(0, r)) = 1.$$

If $V_n$ is the volume of $B(0, 1)$ in $\mathbb{R}^n$, then $V_n r_{\text{eff}}^n = 1 \iff r_{\text{eff}} = V_n^{-\frac{\frac{1}{n}}{\frac{1}{n}}}.$

\underline{Then (Minkowski - Hlawka)}: For any $L \subseteq \mathbb{R}^n$ s.t. $\Upsilon(L) > r_{\text{eff}}$, \ldots
Equivalently: \( \delta_n \geq \frac{c}{2^n} \)

Because: \( \gamma(L) \geq \frac{\mu_{\text{ess}}}{\epsilon \mu_{\text{ess}}} \Rightarrow \exists B \subset L^2 \) where \( B = B(0, \frac{\mu_{\text{ess}}}{\epsilon \mu_{\text{ess}}}) \) are disjoint.

\[
\operatorname{Vol}(B(0, \frac{\mu_{\text{ess}}}{\epsilon})) = \frac{1}{2^n} \operatorname{Vol}(B(0, \mu_{\text{ess}})) = \frac{1}{2^n}.
\]

Remarks: 1. Currently best known bounds on \( \delta_n \) are:

\[
C \frac{n}{2^n} \leq \delta_n \leq (0.67)^n
\]

\[
\downarrow \text{Kolatyanstsi and Levenshtein} \to 8.
\]

Best value of \( C \) due to Venkatesh 2012.

2. History: Minkowski 1911 stated without proof.

Proved by Hlawka '43.

Reproved by Siegel in '45 (apparently filling in details of Minkowski's argument).


No explicit lattices with \( \gamma(L) \geq \epsilon \mu_{\text{ess}} \) are known.
Thm A (Siegel summation formula 145)

There is a measure $m_{\mathbb{R}^n}$ on $\mathbb{R}^n$
which is:
(i) $SL(n, \mathbb{R})$-invariant, regular, Borel.
(ii) a probability measure.
(iii)

$\forall \epsilon \in L^2(\mathbb{R}^n, Vol)$, define $\hat{f}: \mathbb{R}^n \rightarrow \mathbb{R}$

$\hat{f}(L) = \sum_{v \in L \cap \mathbb{Z}^n} f(v)$. $\hat{f}$ is well-defined.

$\forall L \in L^2(\mathbb{R}^n, Vol)$

and

$\int_{\mathbb{R}^n} \hat{f} \, dm_{\mathbb{R}^n} = \int_{\mathbb{R}^n} f \, dVol$. (SSF)

Proof of Minkowski–Hlawka, assuming Thm A

Suppose $M-L$ is false, i.e. for every $L \subseteq \mathbb{R}^n$,

$\forall (L) < \text{rg}$. This means $\forall L \subseteq \mathbb{R}^n$

$LN B(0, \text{rg}) \neq \emptyset$. So $L \cap B(0, \text{rg})$
contains at least two nonzero vectors $v_1, v_2$. 

Let \( f = \frac{1}{B(0, r)} \), by (SSF)

\[
1 = \text{Vol}(B(0, r)) = \int_{\mathbb{R}^n} f \, d\text{Vol} = \int_{\mathbb{R}^n} \frac{1}{r^{n+1}} \, dx_n \geq 2
\]

\[
\hat{f}(L) = \sum_{v \in L \cap \text{Vol}} \frac{1}{B(0, r)}(v) \geq 2
\]

contradiction.

Remark: Can improve const. very slightly by replacing 1 on the LHS with \(2-\varepsilon\).

* If \( A \subset \mathbb{R}^n \) measurable, \( \text{Vol}(A) < 1 \) then \( \exists L \subset \mathbb{R}^n \) s.t. \((L \cup \text{Vol}) \cap A = \emptyset\).

"Reminder" about measure theory.

\((X, \mathcal{B}) \times a \text{ set } B \text{ a } \sigma\text{-algebra} \)

\(B \subset \mathcal{P}(X) \) closed under countable

unions and intersections, complements,
contains \( \emptyset, X \).
a measure \( \mu \) on \((X, \mathcal{B})\) is a function

\[
\mu : \mathcal{B} \to [0, \infty] \\
\mu \left( \bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \mu(A_i), \quad (\Rightarrow A \subseteq B \Rightarrow \mu(A) \leq \mu(B)).
\]

In our setup, \( X \) will be locally compact second countable (lcsc). A special case of lcsc is: \( X \) is a metric space in which closed balls \( B(x, r) = \{ y \in X : d(x, y) < r \} \) are compact. Main example: \( \mathbb{R} \) with the usual metric.

\( \mathcal{B} \) will: Borel \( \sigma \)-algebra - smallest \( \sigma \)-algebra containing the open sets. Then \( \mu \) is called Borel.

\((X, \mathcal{B}, \mu)\) - \( \mu \) is locally finite if

- \( X \) is a locally compact
- there exists for each \( x \in X \) an \( \mathcal{B} \)-measurable \( U \) of \( x \) s.t. \( \mu(U) < \infty \).

In lcsc spaces \( \iff \) \( \mu(K) < \infty \) for any
Compact set $K$.

$\mu$ is called **inner regular** if for any $U$ open set, $\mu(U) = \sup \{\mu(K) : K \subset U \text{ compact}\}$

$\mu$ is called **outer regular** if $\forall S \subset B$,

$\mu(S) = \inf \{\mu(U) : S \subset U \text{ open}\}$

Def $\mu$ is called **Radon** if it is $Borel$, locally finite, inner and outer regular

Let $C_c(X) = \{\text{continuous compactly supported functions } X \to \mathbb{R}\}$

Given a Radon measure $\mu$ on $X$,

define $I_{\mu} : C_c(X) \to \mathbb{R}$

$$I_{\mu}(f) = \int_X f \, d\mu$$

$I_{\mu}$ is a continuous positive linear functional on $C_c(X)$. 
\[ \text{T} \mu(\text{Riesz representation theorem, a.k.a.} \]

Riesz- Kakutani- Markov rep's thm)

The mapping

\[ \{ \text{Radon measures on } X \} \leftrightarrow \{ \text{continuous positive linear functionals on } C_c(X) \} \]

\[ \mu \mapsto \mathcal{I}_\mu \]

is a bijection.

Let \( \psi : C_c(X) \to \mathbb{R} \) be a linear functional.

\( \psi \) is **positive** if \( f \geq 0 \Rightarrow \psi(f) \geq 0 \).

\( \psi \) is **continuous** if for compact \( K \in C > 0 \) s.t. if \( f \) vanishes outside of \( K \) then \( |\psi(f)| \leq C \|f\|_\infty \).

Clearly \( \mathcal{I}_\mu \) is continuous, for each \( K \) take

\[ C = \mu(K) \]
Remark: If $X$ is compact then continuity just means $\Phi$ is a bounded linear functional on $C(X) \leftarrow$ Banach space with sup-norm. Note $C_c(X)$ with sup-norm is not complete when $X$ is not compact. For example $X=\mathbb{R}_\infty$.

\[ f_1 \uparrow h \quad f_2 \downarrow f_3 \]

$f_i$ take values in $[0,1]$, and supported on $[0,1]$

Then $h \cdot f_i \xrightarrow{i \to \infty} h$ in sup-norm but $h \notin C_c(\mathbb{R}_\infty)$.

\[ C_c(X) = \bigcup_{K \subseteq X \text{ compact}} \text{supp } f \subset K^2 \]

$c_c(X)$ is a direct limit of Banach spaces.
With the direct limit topology it becomes a LCTVS.

Cor: In order to define a Radon measure on $X$, suffices to define a continuous pos linear functional on $C(X)$.  
Prop: $C_c(\mathbb{R}^n)$ is dense in $L'(\mathbb{R}^n, Vol)$.

Cor: For any Radon measure $\mu$ on $\mathbb{R}^n$, if (SSF) holds for $\mu$ and for $f \in C_c(\mathbb{R}^n)$, then it holds for all $f \in L'(\mathbb{R}^n)$, and $\hat{f} \in L'(\mu)$.

**Pf:**

$\Psi: C_c(\mathbb{R}^n) \rightarrow L'(\mathbb{R}^n, \mu)$  $\Psi(f) = \hat{f}$

$\hat{f}(L) = \sum_{v \in L \setminus \text{dof}} f(v)$  (note the sum in det

of $\hat{f}$ converges because $L$ is discrete

and $f$ is compactly supported, so sum is finite)
It is continuous as a fn of $L$. (if $B$ is a large ball properly containing support, and $L_j \to L$ (in CF metric)

then for all large enough $j$, $B \times \text{ball}$

exists $x_j, y_j \in B$, $x_j \to x$, and $B(\delta) = x_j y_j$.

and so the sum in def. of $\hat{f}(L)$,

$\hat{f}(L_j)$ is finite with a bound on the number of nonzero summands).

We are assuming

\[ \lim_{\mu \to \infty} \int_{\mathbb{R}^n} |f| \, d\mu = \int_{\mathbb{R}^n} \hat{f} \, d\text{vol} \quad \forall f \in C_c(\mathbb{R}^n) \]

\[ \| \psi(f) \|_{L^1(\mu)} = \int_{\mathbb{R}^n} |\hat{f}| \, d\mu = \int_{\mathbb{R}^n} (1+|\xi|^2)^{\frac{1}{2}} \, d\mu = \int_{\mathbb{R}^n} \psi(\hat{f}) \, d\mu \]

which is finite $L^1(\text{vol})$ (SSF)

Similarly (check!) $\| \psi(f) \|_{L^1(\mu)} = \| f \|_{L^1(\text{vol})}$
\[ \psi, \phi \text{ are linear functionals on } C_c(\mathbb{R}^n) \]
which are \( L^1 \)-banded. Since \( C_c(\mathbb{R}^n) \) is
dense in \( L^1(\mathbb{R}^n, \text{Vol}) \), there is a unique
continuous extension of \( \psi, \phi \) to all \( f \in L^1(\mathbb{R}^n, \text{Vol}) \).

It can be checked (ex.) that this extension
of \( \psi, \phi \) to \( L^1(\mathbb{R}^n, \text{Vol}) \) is given by
the same formula.

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**Radon-Nikodym Theorem**

Let \( \mu_1, \mu_2 \) be Radon measures on \((\mathbb{R}^n, \mathcal{X})\).

We say \( \mu_1 \) is absolutely continuous w.r.t. \( \mu_2 \)
(abbreviation \( \mu_1 \ll \mu_2 \)) if
\[
\forall A \in \mathcal{B}, \mu_2(A) = 0 \implies \mu_1(A) = 0.
\]

Then (Radon-Nikodym 1) if \( \mu_1 \ll \mu_2 \) as above
then \( \exists \) measurable \( \psi : X \to \mathbb{R}_{\geq 0} \) s.t. for
\[
\forall A \in \mathcal{B}, \mu_1(A) = \int_A \psi \, d\mu_2 \quad (\star)
\]
\[ \psi \text{ is unique up to zero measure, i.e., if } \psi, \psi' \text{ satisfy } (4) \text{ then } \psi(x) = \psi'(x) \text{ for } \mu\text{-a.e. } x. \]

Notation: \[ \psi(x) = \frac{d\psi}{d\mu} (x). \]

Example: When \( X = \mathbb{R}^N \), \( \mu = \text{Vol} \), and \( \mathbf{g} \in \text{GL}(\mathbb{R}) \), \( \nu_2(A) = \mu(g^{-1}(A)) \), then

\[ \frac{d\psi}{d\mu} = |\det(\mathbf{g})|; \quad (= \text{Jacobian formula}). \]

\[ \text{Recall: } G \text{ a group, } X \text{ a set.} \]

An action of \( G \) on \( X \) (notation \( G \times X \))

is a map \( G \times X \to X \) (action map)

\[ (g, x) \mapsto g \cdot x \]

s.t. \( g_1(g_2x) = (g_1g_2)x \) and \( e \cdot x = x \) \( \forall x \in X \)

\[ (g_1g_2) \cdot x = g_2 \cdot (g_1 \cdot x) \]

In our setup, \( G \) and \( X \) will be Lie groups, and
action map is required to be continuous.

Example: $G = GL_n(\mathbb{R})$, $d(g_1, g_2) = \|g_1 - g_2\|_{op}$

Any closed subgroup is lsc.

If $G$ acts on $X$, it also acts on Radon measures on $X$ by $g_* \mu(A) = \mu(g^{-1}(A))$ where $g \in G$, $A \in \mathcal{B}$, $\mu$ is Radon.

$g_* \mu$ is called the pushforward of $\mu$ by $g$.

In terms of R.R.T.,

$$\int_X f dg_* \mu = \int_X (f \circ g) d\mu$$

The identity $\int_X f dg_* \mu = \int_X f d\mu$ holds for all $f \in L^1(g_* \mu)$.

Def $\mu$ is $G$-invariant if $g_* \mu = \mu$ for all $g \in G$. 
GQX is called \textit{transitive} if \( \forall x_1, x_2 \in X \exists g \in G \text{ s.t. } g(x_1) = x_2 \).

\textbf{Prop.} If \( GQX \) transitive, then any two nonzero \( G \)-inv. Radon measures are proportional, i.e., if \( \mu_1, \mu_2 \) are both \( G \)-inv. and Radon, and nonzero, then \( \exists c > 0 \text{ s.t. } \forall A \in B, c \mu_1(A) = \mu_2(A) \).

\textit{Proof (sketch, details in ex.):}

Assume \( \mu_1 \ll \mu_2 \). Let \( \psi = \frac{d\mu_1}{d\mu_2} \).

Using the \( G \)-invariance of \( \mu_1, \mu_2 \) can show that \( \psi \) is constant a.e.

More precisely, after modifying \( \psi \) on a subset of measure 0 w.r.t. \( \mu_2 \), \( \psi \) is \( G \)-inv. (exercise).
By transitivity, \( \phi \) is constant.

\[ \Rightarrow c \mu_1 = \mu_2 \quad \text{where } c \text{ is the constant value of } \phi. \]

If \( \mu_1 \) is not \( \alpha \)-c. w.r.t. \( \mu_2 \), define \( \mu_3 = \mu_1 + \mu_2 \). Now \( \mu_1 \ll \mu_3 \) i.e.,

By previous case, \( \exists \) constants \( c_1, c_2 \) s.t.

\[ \mu_3 = c_1 \mu_1 = c_2 \mu_2 \Rightarrow \mu_1, \mu_2 \text{ are proportional.} \]

**Examples**

1. \( X = \mathbb{R}^n, G = \mathbb{R}^n \)

   \( G \) acts by translations.

   \( \text{Vol} \) is the unique (up to scaling)

   \( G \)-invar. Borel measure on \( \mathbb{R}^n \).

2. \( X = \mathbb{R}^n, G = \text{GL}(\mathbb{R}) \), action by

   linear transformations.

   \( G \cap \mathbb{R}^n \text{--dof, } G_0 \text{--dof.} \)

   On \( \mathbb{R}^n \text{--dof can use } \text{Vol}. \)
By Jacobi's formula, $G$ preserves $Vol$ on $R^n$.

On $\mathfrak{g}_0$, $\delta_0$ (Dirac measure on $\mathfrak{g}_0$) is $G$-invariant. Therefore, any $G$-invariant measure on $R^n$ is of the form $c Vol + \delta_0$.

(2) $G = SL_n(12)$ (or any other group) acts on itself by left and right translations,

$$L_g(g_0) = g g_0, \quad R_g(g_0) = g^{-1} g_0$$

$$L_g \circ L_{g_2} = L_{g_1 g_2}, \quad R_g \circ R_{g_2} = R_{g_1 g_2}$$

Given $A \subset SL_n(12)$ (Borel), define $C(A) = \{ x : a \in A, \text{ for } x \}$ ("the cone over $A$")

\[ A \quad \text{SL}_{n(12)} \]
\[ C(A) \quad 0 \in M_n(12) \]
Define \( m_{SL_n(\mathbb{R})}(A) = \text{Vol}_{M_n(\mathbb{R})}(C(A)) \)
\( (M_n(\mathbb{R}) = \mathbb{R}^{n^2}) \)

Claim: \( m_{SL_n(\mathbb{R})} \) is inv. under both \( L_g \) and \( R_g \) for any \( g \in SL_n(\mathbb{R}) \).

Proof: Note if \( g \in G \), 
\[ C(L_g(A)) = gC(A) \]
\[ C(R_g(A)) = C(A)g^{-1} \]

So to prove claim, suffices to show that for any \( S \subset M_n(\mathbb{R}) \) (Borel), \( g \cdot S \cdot g^{-1} \).

\[ \text{Vol}_{M_n(\mathbb{R})}(gS) = \text{Vol}_{M_n(\mathbb{R})}(S) = \text{Vol}_{M_n(\mathbb{R})}(Sg) \]

Hence, write a matrix \((v_1, \ldots, v_n)\) 
\[
= \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^{n^2}
\]

In these coordinates, \( L_g \) acts on \( M_n(\mathbb{R}) \) by
\[

g \begin{pmatrix} g \\ g \\ \vdots \\ g \end{pmatrix} \]

The Jacobian of \( L_g : M_n(\mathbb{R}) \rightarrow M_n(\mathbb{R}) \)
$$(\det g)^n = 1 \quad \text{because } g \in \text{SL}_n(\mathbb{R}).$$

Similarly, $R_g$ preserves $\operatorname{Vol}(\text{Mat}(n,\mathbb{R}))$ and hence preserves $\mu_{\text{SL}_n(\mathbb{R})}$.

**Remark:** If $G$ is locally compact, it has a right Haar measure and a left measure $\nu_R$ and $\nu_L$.

They are the unique (up to proportionality) measures on $G$ which are $R_g$ and $L_g$-invariant, for any $g \in G$.

If $\nu_L = \nu_R$ (up to proportionality) then $G$ is called unimodular. We exhibited a measure on $\text{SL}_n(\mathbb{R})$ which is left- and right-invariant, hence is Haar measure, and $\text{SL}_n(\mathbb{R})$ is unimodular.

$$(\circ) \quad U = \{ g \in \text{SL}_n(\mathbb{R}) : ge_i = e_i \} \quad e_i = (0^i, 1, \ldots, 0^n).$$
\[ U = \left\{ \begin{pmatrix} 1 & x \\ 0 & A \end{pmatrix} : A \in \mathbb{R}^{n \times n}, x \in \mathbb{R} \right\} \]

\[ m_\mu(S) = \sum_{A \in \mathbb{R}^{n \times n}} \int_S \frac{1}{\operatorname{det}(A)} \, d\operatorname{vol}(x) \, dm_{\mathbb{R}^{n-1}}(A) \]

\[ \text{mu is both left and right invariant.} \]

To see this, let's compute the group law on \( U \): Note that \((A_0, x_0)(A_2, x_2) = (A_0 A_2, x_0 + x_2 A_2)\) (this is an example of a semi direct product).

\[ \int_{U} f \circ L(A_0, x_0)(A_2, x_2) \, dm_{\mu}(A, x) = \]

\[ = \sum_{A \in \mathbb{R}^{n \times n}} \int_{\mathbb{R}^{n-1}} f(A_0 A, x + x_0 A) \, d\operatorname{vol}(x) \, dm_{\mathbb{R}^{n-1}}(x) \, dm_{\mathbb{R}^{n-1}}(A) \]
\[ = \int_{\text{Vol is inv. in trans. by } x_0 A} \int_{\text{Vol is inv. in trans. by } x_0 A} f(A_0 A) \, d\text{Vol}_{\mathbb{R}^{n-1}(x)} \, d\mu_{\text{SL}(n-1)(\mathbb{R})}(A) \]

Similarly for right translation:

\[ (A_0 x_0) = (A_0^{-1}, -x_0 A_0^{-1}) \]

\[ \int_{\text{Vol is inv. in trans. by } x_0 A} \int_{\text{Vol is inv. in trans. by } x_0 A} f(A_0 A) \, d\text{Vol}_{\mathbb{R}^{n-1}(x)} \, d\mu_{\text{SL}(n-1)(\mathbb{R})}(A) \]

\[ = \int_{\text{Vol is inv. in trans. by } x_0 A} \int_{\text{Vol is inv. in trans. by } x_0 A} f(A_0 A_0^{-1} x_0 A_0^{-1}) d\text{Vol}_{\mathbb{R}^{n-1}(x)} d\mu_{\text{SL}(n-1)(\mathbb{R})}(A) \]

\[ = \int_{\text{Vol is inv. in trans. by } x_0 A} \int_{\text{Vol is inv. in trans. by } x_0 A} f(A_0 A_0^{-1} x_0 A_0^{-1}) d\text{Vol}_{\mathbb{R}^{n-1}(x)} d\mu_{\text{SL}(n-1)(\mathbb{R})}(A) \]
Volume is an even function under right multiplication by $A_n$ since $\det(A_n x) = 1$.

$$ \text{Vol}_{\mathbb{R}^n} = \int \int f(A, x) \, d\text{Vol}_{\mathbb{R}^n}(x) \, d\mu_{\mathbb{R}^n}(A) $$

$$ = \int f(A, x) \, d\mu_n(A, x) $$

\( \text{(5) The affine group} \)

$$ \text{ASL}_n(\mathbb{R}) = \left\{ \begin{pmatrix} A & x \\ 0 & 1 \end{pmatrix} : A \in \text{SL}_n(\mathbb{R}), x \in \mathbb{R}^n \right\} $$

\( \text{Group law: } \ (A_1, x_1)(A_2, x_2) = \begin{pmatrix} A_1 & x_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_2 & x_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A_1 A_2 & x_1 + A_1 x_2 \\ 0 & 1 \end{pmatrix} \)

Define $\mu_{\text{ASL}_n(\mathbb{R})}$ by setting $\mu_{\text{ASL}_n(\mathbb{R})}(A)$
\[
\int_{\text{ASL}(\mathbb{C})} f(A, x) \, d\mu_{\text{ASL}(\mathbb{C})} = \int_{\text{SL}(n)} \int_{\mathbb{R}^n} f(A, x) \, d\mu_{\text{SL}(n)}(x) \, d\mu_{\text{ASL}(\mathbb{C})}(A).
\]

A similar computation to the one in (4) shows that $M_{\text{ASL}(\mathbb{C})}$ is both left- and right-invariant.