Example of lcsc: a metric space containing a dense countable set (separable), s.t. $\forall x \; \exists r_0 > 0 \; \forall r \in (0, r_0)$

$\bar{B}(x,r) = \{x' \in X : d(x,x') \leq r\}$ is compact. Ex. $\mathcal{X}_n$ has these properties

Reminders from lecture 1.

Fundamental domain

Let $L \triangleleft \mathbb{R}^n$ be a lattice.

Def A set $\Omega \subset \mathbb{R}^n$ is a fundamental domain for L if: (i) Ω is a Borel set.

(ii) For every $x \in \mathbb{R}^n$ there is a unique $y \in \Omega$ s.t. there is $\ell \in L$ with $y = x - \ell$.

Restatements of (ii):

- $\displaystyle\bigsqcup_{\ell \in L} \ell + \Omega = \mathbb{R}^n$

disjoint union

- Ω is a collection of equivalence class representatives for the relation $x_1 \sim x_2 \iff x_1 - x_2 \in L$
- Ω is a collection of coset representatives for the quotient $\mathbb{R}^n / L$.

Prop If A and B are two fundamental domains for L then $\mathrm{Vol}(A) = \mathrm{Vol}(B)$.

Pf For each $b \in B$, define $\ell \in L$, $\ell = \ell(b)$,

and $a \in A$, $a = a(b)$ by the requirement $a = b - \ell$. (by (ii) this is well-defined).

Define, for $\ell_0 \in L$, $B_{\ell_0} = \{ b \in B : \ell(b) = \ell_0 \}$.

B_{ℓ_0} is a Borel set. Because

$$B_{\ell_0} = B \cap (A + \ell_0).$$

By uniqueness in (ii), $B = \bigsqcup_{\ell \in L} B_\ell$

$$A = \bigsqcup_{\ell \in L} B_\ell - \ell.$$

So $\mathrm{Vol}(B) = \sum_{\ell \in L} \mathrm{Vol}(B_\ell) = \sum_{\ell \in L} \mathrm{Vol}(B_\ell - \ell)$

$$= \mathrm{Vol}(A).$$

Geometry of Numbers lecture 9

Recall from last time: * lcsc space - locally compact second countable $\Leftarrow$ separable (contains a dense countable set), metric space, $\forall x \in X\ \exists r$ s.t. $\overline{B}(x,r) = \{y \in X : d(x,y) \leq r\}$ is compact.

* Radon measure: defined on Borel σ-algebra regular and locally finite

<u>Riesz rep'n thm</u>: $\{$Radon measures on $X\} \longleftrightarrow \{$pos. cts. linear functionals on $C_c(X)\}$

<u>Prop 1</u> G, X lcsc $G \curvearrowright X$ transitively, then a G-inv. Radon measure on X is unique up to scaling (if μ_1, μ_2 are G-inv. Radon measures nonzero, then $\exists c > 0$ s.t. $\forall A \in \mathcal{B}$, $\mu_1(A) = c\mu_2(A)$).

Every lcsc group G has a right-invariant Radon measure and a left-inv. Radon measures.

These are the right and left Haar measures on G.
If a right-Haar measure is also left-invariant,
G is called unimodular.

<u>Examples</u>: * G abelian.

* $G = SL_n(\mathbb{R})$

* $U = \{g \in G : g e_1 = e_1\} = \left\{ \left(\begin{array}{c|c} 1 & * \\ \hline 0 & \\ \vdots & * \\ 0 & \end{array} \right) \right\}$.

* $ASL_n(\mathbb{R}) = \left\{ \left(\begin{array}{c|c} * & * \\ \hline 0 \cdots 0 & 1 \end{array} \right) \right\} < SL_{n+1}(\mathbb{R})$

Prop 2 For these three examples, we gave Haar measure explicitly and showed it is both left- and right-invariant.

<u>Next goal</u>: Construct a G-inv. Radon measure on $\mathcal{X}_n = G/SL_n(\mathbb{Z})$.

More generally: for a unimodular lcsc group H, and a discrete subgroup Γ, want to construct an H-inv. measure on H/Γ.

<u>Def:</u> Suppose H lcsc, Γ a closed subgroup, then $\Omega \subset H$ is called <u>a fundamental domain</u> for H/Γ if: $*$ Ω is Borel set

$*$ $\forall h \in H$ $\exists!$ $w \in \Omega \cap h\Gamma$

$\Updownarrow$

If $\pi: H \longrightarrow H/\Gamma$, then $\pi|_{\Omega}: \Omega \to H/\Gamma$ is a bijection.

<u>Prop:</u> For H lcsc, Γ discrete, a fundamental domain for H/Γ exists.

<u>Pf:</u> We will use the following fact: any lcsc group H has a right-inv. metric which induces the topology.

(right-invariant: $\forall h_1, h_2, h_3 \in H$, $d(h_1, h_2) = d(h_1 h_3, h_2 h_3)$).

For G or a subgroup of G, can use

$d(g_1, g_2) = \| g_1 g_2^{-1} - Id \|_{op}$ (check!)

By discreteness of Γ, for any $h \in H$ $\exists r > 0$ s.t. $\pi|_{B(h,r)}$ is injective.

(if not, $\exists h_i \to h$, $\gamma_i \in \Gamma$ s.t. $h_i \gamma_i \to h$
$\gamma_i \neq e$

$\gamma_i = \underbrace{h_i^{-1}}_{h^{-1}} \underbrace{h_i \gamma_i}_{h} \longrightarrow e$, $\gamma_i \neq e$, Γ discrete contradiction)

Define $r_h = \sup \{ r > 0 : \pi|_{B(h,r)} \text{ is injective} \}$.

Note: If $h_1 \in B(h, \frac{r_h}{2})$ then $r_{h_1} \geq \frac{r_h}{2}$.
(because $B(h_1, \frac{r_h}{2}) \subset B(h, r_h)$).

Let $H_0 \subset H$ be countable and dense

Then $\bigcup_{h_0 \in H_0} B(h_0, \frac{r_{h_0}}{2}) = H$.

because: if $h \in H$, and $h_0 \in B(h, \frac{r_h}{4})$ then $h \in B(h_0, \frac{r_h}{4}) \subset B(h_0, \frac{r_{h_0}}{2})$.

Write $H_0 = \{ h_1, h_2, h_3, \dots \}$

$B_i = B(h_i, \frac{r_{h_i}}{2})$. π injective on each B_i.

Define $\Omega = B_1 \cup (B_2 \setminus B_1\Gamma) \cup B_3 \setminus ((B_1 \cup B_2)\Gamma) \cup \dots$

$$= \bigcup_{i=1}^{\infty} \left(B_i \setminus (\bigcup_{j<i} B_j)\Gamma \right).$$

Then $\pi|_\Omega$ is a bijection.

Prop 4: Suppose H is lcsc and unimodular, with Haar measure m_H, $\Gamma < H$ discrete.

Then: (1) any two fund. domains for H/Γ have the same measure (w.r.t. m_H)

(2) If A, B are Borel subsets of H with $\pi(A) = \pi(B)$, $\pi|_A$ and $\pi|_B$ are injective, then $m_H(A) = m_H(B)$.

(3) If $I \subset J$, I, J Borel subsets of H, $\pi|_I$ injective, $\pi|_J$ surjective, then there is a fund. domain Ω for H/Γ s.t.

$$I \subset \Omega \subset J.$$

Pf of (2) (Note (1) is a special case of (2), (3) is left as an exercise).

For each $a \in A$ there is $\gamma_a \in \Gamma$ s.t. $a\gamma_a \in B$. γ_a is unique (because $\pi|_B$ is injective).

Define $A_\gamma = \{a \in A : \gamma_a = \gamma\}$, $\gamma \in \Gamma$

$$A = \bigsqcup_{\gamma \in \Gamma} A_\gamma, \quad B = \bigsqcup_{\gamma \in \Gamma} A_\gamma \gamma$$

$$m_H(A_\gamma) = m_H(A_\gamma \gamma) \quad (m_H \text{ is right-inv.}).$$

$$m_H(A) = \sum_{\gamma \in \Gamma} m_H(A_\gamma) = \sum_{\gamma \in \Gamma} m_H(A_\gamma \gamma) = m_H(B).$$

Prop 5: Let H be lcsc unimodular. $\Gamma \subset H$ discrete, Ω a fundamental domain for H/Γ. Define a measure $m_{H/\Gamma}$ on H/Γ by $m_{H/\Gamma}(A) = m_H(\Omega \cap \pi^{-1}(A))$, A a Borel subset of H/Γ

Then $m_{H/\Gamma}$ is well-defined and inv. under left-translation by elements of H.

Pf: Well-defined by Prop 4, (2).

Left invariance: if $h \in H$, $A \subset H/\Gamma$ Borel, then

$$m_{H/\Gamma}(hA) = m_H(\Omega \cap \pi^{-1}(hA))$$

$$= m_H(\Omega \cap h\pi^{-1}(A))$$

$$= m_H(h\Omega \cap h\pi^{-1}(A)) = m_H(h(\Omega \cap \pi^{-1}(A)))$$

$$= m_H(\Omega \cap \pi^{-1}(A)) = m_{H/\Gamma}(A).$$

Ω a fund. domain $\iff$ $h\Omega$ fund domain, by well-definedness

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We now have measures $m_{\mathcal{X}_n}$ on $\mathcal{X}_n = G/SL_n(\mathbb{Z})$ as well as on $U/U_{\mathbb{Z}}$.

Def: H lcsc, Γ discrete, H unimodular. Γ is called <u>a lattice in H</u> if

$m_H(H/\Gamma)$ is finite $\iff m_H(\Omega) < \infty$ for a fund. domain Ω for H/Γ.

Example: any lattice $L \subset \mathbb{R}^n$ (in previous def'n) is a lattice in $\mathbb{R}^n$ (with new def'n).

Recall: Thm (Siegel): (i) $SL_n(\mathbb{Z})$ is a lattice in G.

(ii) For $f \in L^1(\mathbb{R}^n; \mathrm{Vol})$, define $\hat{f} : \mathcal{X}_n \to \mathbb{R}$

by $\hat{f}(L) = \sum_{v \in L \setminus \{0\}} f(v)$.

Then (SSF) $\int_{\mathcal{X}_n} \hat{f}\, dm_{\mathcal{X}_n} = \int_{\mathbb{R}^n} f\, d\mathrm{Vol}$. (where $m_{\mathcal{X}_n}$ normalized by $m_{\mathcal{X}_n}(\mathcal{X}_n) = 1$).

Last time we showed that it suffices to prove (SSF) for $f \in C_c(\mathbb{R}^n \setminus \{0\})$.

Today we will prove:

Thm: ① $U_{\mathbb{Z}}$ is a lattice in $\mathcal{U}$

② $\exists\, c_n > 0$ s.t. if we define, for $f \in L^1(\mathbb{R}^n; \mathrm{Vol})$,

$$\hat{f}^{P}(L) = \hat{f}(L) = \sum_{\substack{v \in L \\ v \text{ primitive}}} f(v) \quad , \quad \text{then}$$

$$c_n \int_{\mathcal{X}_n} \hat{f} \, dm_{\mathcal{X}_n} = \int_{\mathbb{R}^n} f \, d\mathrm{Vol} \quad \left(m_{\mathcal{X}_n}(\mathcal{X}_n) = 1\right).$$

(3) $SL_n(\mathbb{Z})$ is a lattice in G.

We will work with the following diagram.

$$\begin{tikzcd}
 & G/U_{\mathbb{Z}} = \mathcal{P}_n = \left\{ (v, L) : L \in \mathcal{X}_n, \ \begin{array}{l} v \in L \\ v \text{ primitive} \end{array} \right\} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
G/U = \mathbb{R}^n \setminus \{0\} & & \mathcal{X}_n = G/SL_n(\mathbb{Z})
\end{tikzcd}$$

fiber is $U/U_{\mathbb{Z}}$

fiber is $SL_n(\mathbb{Z})/U_{\mathbb{Z}}$

$\pi_1(v, L) = v$

$\pi_2(v, L) = L$.

Let's understand the spaces in the diagram as quotients of groups.

G acts transitively on $\mathbb{R}^n \setminus \{0\}$, $\mathrm{stab}_G(e_1) = U$

so $\mathbb{R}^n \setminus \{0\} = G/U$.

$\mathcal{X}_n = G/SL_n(\mathbb{Z})$.

<u>Note:</u> G acts transitively on $\mathcal{P}_n$. Because:

Suppose (L, v), (L', v'), want $g \in G$

s.t. $gL = L'$, $gv = v'$. WNLG that $L' = \mathbb{Z}^n$.

Since $G \curvearrowright \mathcal{X}_n$ is transitive, $\exists g \in G$ s.t.

$gL = \mathbb{Z}^n$. And $gv = v''$ is primitive in $\mathbb{Z}^n$.

We saw $SL_n(\mathbb{Z})$ acts transitively on

primitive vectors in $\mathbb{Z}^n$. Hence $\exists \gamma \in SL_n(\mathbb{Z})$

s.t. $\gamma v'' = v'$, and $\gamma \mathbb{Z}^n = \mathbb{Z}^n$.

So $\gamma g L = \mathbb{Z}^n$, $\gamma g v = v'$. $(\gamma g \in G)$.

$\mathrm{Stab}_G((e_1, \mathbb{Z}^n)) = U \cap SL_n(\mathbb{Z}) = U_{\mathbb{Z}}$

$U/U_{\mathbb{Z}}$ is the space of all lattices in $\mathcal{X}_n$ which contain e_1 as a primitive vector (check!).

To use the diagram, we will consider two operations on functions.

Given $f : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}$, define

$\bar{f} : P_n \longrightarrow \mathbb{R} \qquad \bar{f} = f \circ \pi_1$

Given $h : P_n \longrightarrow \mathbb{R}$ define $h^* : \mathcal{X}_n \to \mathbb{R}$

$$h^*(L) = \sum_{y \in \pi_2^{-1}(L)} h(y) = \sum_{\substack{v \in L \\ v \text{ primitive}}} h(v, L).$$

This is called "folding" or "periodization".

Remark: 1. Periodization is common in Fourier analysis.
2. In our notations, $\hat{f}(L) = (\bar{f})^*(L)$.

The maps $f \mapsto \bar{f}$, $h \mapsto h^*$ are linear and positive

Lemma 1 (folding identity) $\exists c_n' > 0$ s.t. $\forall h \in L^1(\mathcal{P}_n, m_{\mathcal{P}_n})$

$$c_n' \int_{\mathfrak{X}_n} h^* \, dm_{\mathfrak{X}_n} = \int_{\mathcal{P}_n} h \, dm_{\mathcal{P}_n}. \qquad (*)$$

Pf. Suffices to prove $(*)$ for $h \in C_c(\mathcal{P}_n)$.
($C_c(\mathcal{P}_n)$ is dense in $L^1(\mathcal{P}_n, m_{\mathcal{P}_n})$ and $(*)$ defines a bounded linear functional).

Define a Radon measure μ on $\mathcal{P}_n$ by

$$\int_{\mathcal{P}_n} h \, d\mu = \int_{\mathfrak{X}_n} h^* \, dm_{\mathfrak{X}_n}.$$

$\operatorname{supp}(h^*) \subset \pi_2(\operatorname{supp} h)$, $h \in C_c(\mathcal{P}_n)$

$\Longrightarrow h^* \in C_c(\mathfrak{X}_n) \Longrightarrow \int_{\mathfrak{X}_n} h^* \, dm_{\mathfrak{X}_n}$ exists.

So we've defined a pos. linear functional on $C_c(\mathcal{P}_n)$, hence a measure.

Check that μ is G-inv.:

$$h \circ g(y) = h(gy)$$

$$\Rightarrow (h \circ g)^*(L) = \sum_{y \in \pi^{-1}(L)} h(gy) \underset{z = gy}{=} \sum_{z \in \pi^{-1}(gL)} h(z) = h^*(gL)$$

So
$$\int_{\mathcal{X}_n} (h \circ g)^* \, dm_{\mathcal{X}_n} = \int_{\mathcal{X}_n} h^* \, dm_{\mathcal{X}_n}$$
$m_{\mathcal{X}_n}$ is G-inv.

So μ is G-inv. By Prop 1, there is a unique

G-inv. Radon measure on $\mathcal{P}_n$, namely $m_{\mathcal{P}_n}$.

So $\exists\, c_n'$ s.t. $c_n' \mu = m_{\mathcal{P}_n}$, this

implies $(*)$.

<u>Proof of Thm</u> By induction on n, starting with $n=2$.

① $U = \left\{ \left(\begin{array}{c|c} 1 & x \\ 0 & A \end{array}\right) : \begin{array}{l} x \in \mathbb{R}^{n-1} \\ A \in SL_{n-1}(\mathbb{R}) \end{array} \right\} =$

$\underset{n=2}{=} \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} : s \in \mathbb{R} \right\} \cong \mathbb{R}$

$U_{\mathbb{Z}} = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\} \cong \mathbb{Z}.$

This proves ① (for $n=2$).

② We will show π_1 is <u>proper</u>,

i.e. the preimage of a compact set is compact

<u>Remark:</u> This is only true for $n=2$, not $n \geq 3$!

<u>Heuristic</u> fiber of π_1 is $U/U_{\mathbb{Z}}$ which is compact for $n=2$.

Let $K \subset \mathbb{R}^2 \setminus \{0\}$ be compact, choose $K_0 \subset G$ be compact so that $K \subset K_0 e_1$

Let $U_0 = \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} : s \in [0,1] \right\} \subset U$
compact

$U_0 U_{\mathbb{Z}} = U$.

Then $K_0 U_0 \subset G$ is compact.

$K_0 U_0 / U_{\mathbb{Z}} \subset \mathcal{P}_2$ contains $\pi_1^{-1}(K)$.

So $\pi_1^{-1}(K)$ is compact.

Now to obtain ②, repeat the arguments as in proof of Lemma 1.

Given $f \in L^1(\mathbb{R}^n)$. Suffices (SSF) for $f \in C_c(\mathbb{R}^n \setminus \{0\})$. By properness, $\bar{f} \in C_c(\mathcal{P}_2)$ and hence (as in Lemma 1) $\hat{f} \in C_c(\mathcal{X}_2)$

Now define μ on $\mathbb{R}^2 \setminus \{0\}$, a Radon measure, by $\int_{\mathbb{R}^2 \setminus \{0\}} f \, d\mu = \int_{\mathcal{X}_2} \hat{f} \, dm_{\mathcal{X}_2}$.

Well-defined by Riesz rep'n thm and since $\hat{f} \in C_c(\mathcal{X}_2)$. As before, μ is G-inv. on $\mathbb{R}^2 \setminus \{0\}$. By Prop 1 (uniqueness) μ is a multiple of Vol.

③ We will see ② $\Rightarrow$ ③ for all n.

We have (SSF) for all $f \in L^1(\mathbb{R}^n, \mathrm{Vol})$.

Let $f = \mathbb{1}_B$ $\quad B = [-1.1, 1.1]^n$. $\mathrm{Vol}(B) > 2^n$.

For any $L \in \mathcal{X}_n$,

$\hat{f}(L) = \#\{v \in L : v \in L, v \text{ primitive}\}$

≥ 1

$\uparrow$

By Minkowski's 1st thm.

$$m_{\mathcal{X}_n}(\mathcal{X}_n) = \int_{\mathcal{X}_n} 1 \cdot dm_{\mathcal{X}_n} \leq \int_{\mathcal{X}_n} \hat{f}(L)\, dm_{\mathcal{X}_n}(L)$$

$$= \frac{1}{c_n} \int_{\mathbb{R}^n} f \, d\mathrm{Vol} = \frac{\mathrm{Vol}(B)}{c_n} < \infty .$$

General case, $n \geq 3$: ① We had a formula for Haar measure on $\mathcal{U}$: For $S \subset \mathcal{U}$

$$m_{\mathcal{U}}(S) = \int_{SL_{n-1}(\mathbb{R})} \int_{\mathbb{R}^{n-1}} \mathbb{1}_S \left(\begin{array}{c|c} 1 & x \\ \hline 0 & A \end{array} \right) d\mathrm{Vol}(x) \, dm_{SL_{n-1}(\mathbb{R})}(A)$$

Let Ω be a fundamental domain for $SL_{n-1}(\mathbb{R}) / SL_{n-1}(\mathbb{Z})$, then $m_{SL_{n-1}(\mathbb{R})}(\Omega) < \infty$ (by induction hypothesis, ③ for $n-1$).

Claim: $\Omega_{\mathcal{U}} = \left\{ \left(\begin{array}{c|c} 1 & x \\ \hline 0 & A \end{array} \right) : \begin{array}{l} A \in \Omega \\ x \in [0,1)^{n-1} \end{array} \right\}$

contains a fund. domain for $\mathcal{U} / \mathcal{U}_{\mathbb{Z}}$.

(actually it is a fund. domain, but we don't need this).

Indeed, given $\left(\begin{array}{c|c} 1 & x \\ \hline 0 & A \end{array}\right) \in \mathcal{U}$

let $\gamma_1 \in SL_{n-1}(\mathbb{Z})$ s.t. $A\gamma_1 \in \Omega$

let $\gamma_2 = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \gamma_1 \end{array}\right) \in \mathcal{U}_{\mathbb{Z}}$.

$$\left(\begin{array}{c|c} 1 & x \\ \hline 0 & A \end{array}\right) \gamma_2 = \left(\begin{array}{c|c} 1 & y \\ \hline 0 & A\gamma_1 \end{array}\right)$$

let $\gamma_3 = \left(\begin{array}{c|c} 1 & p \\ \hline 0 & \mathbb{1}_{n-1} \end{array}\right) \in \mathcal{U}_{\mathbb{Z}}$, $p \in \mathbb{Z}^{n-1}$, $y+p \in [0,1)^{n-1}$

$$\left(\begin{array}{c|c} 1 & x \\ \hline 0 & A \end{array}\right) \gamma_2 \gamma_3 = \left(\begin{array}{c|c} 1 & y+p \\ \hline 0 & A\gamma_1 \end{array}\right) \in \Omega_{\mathcal{U}}.$$

This proves claim.

$$m_{\mathcal{U}}(\Omega_{\mathcal{U}}) = \int_{SL_{n-1}(\mathbb{R})} \mathbb{1}_{\Omega}(A)\, dm_{SL_{n-1}(\mathbb{R})}(A)$$

$$= m_{SL_{n-1}(\mathbb{R})}(\Omega) < \infty.$$

(2) To prove (SSF), suffices as before to take $f \in C_c(\mathbb{R}^n \setminus \{0\})$.

<u>We claim</u> $\bar{f} = f \circ \pi_1 \in L^1(\mathcal{P}_n, m_{\mathcal{P}_n})$.

We will write $m_{\mathcal{P}_n}$ in a different way (not as m_G restricted to a fund. domain).

By Prop 1, $m_{\mathcal{P}_n}$ is the unique G-inv measure on $\mathcal{P}_n$. The set $\pi_1^{-1}(e_1) =$

$$= \{(e_1, L) : L \in \mathcal{X}_n,\ e \in L \text{ is primitive}\}$$

$\cong U / U_{\mathbb{Z}}$. (U acts transitively on $\pi_1^{-1}(e_1)$ and $U_{\mathbb{Z}}$ preserves $(e_1, \mathbb{Z}^n)$).

Let ν_{e_1} be the image of $m_{U/U_{\mathbb{Z}}}$ under the map identifying $U/U_{\mathbb{Z}}$ with $\pi_1^{-1}(e_1)$.

By ① ν_{e_1} is a finite measure, and we normalize it to be a probability measure. For each $y \in \mathbb{R}^n \setminus \{0\}$ define a probability measure ν_y on $\mathcal{P}_n$ supported on $\pi_1^{-1}(y)$, $\nu_y = g_* \nu_{e_1}$ where $ge_1 = y$.

This only depends on y (ind. of choice of g) because if $g_1 e_1 = y = g_2 e_1$, then $g_1^{-1} g_2 e_1 = e_1$, so $g_1^{-1} g_2 \in U$, so

$$(g_1^{-1} g_2)_* \nu_{e_1} = \nu_{e_1} \Rightarrow (g_2)_* \nu_{e_1} = (g_1)_* (g_1^{-1} g_2)_* \nu_{e_1} = (g_1)_* \nu_{e_1}$$

$(*)$

Define μ on $\mathcal{P}_n$ by $\forall f \in C_c(\mathcal{P}_n)$

$$\int_{\mathcal{P}_n} f \, d\mu = \int_{\mathbb{R}^n \setminus \{0\}} \int_{\mathcal{P}_n} f \, d\nu_y \, d\mathrm{Vol}(y) \quad (**)$$

Well-defined because $y \mapsto \int_{\mathcal{P}_n} f \, d\nu_y$

is continuous and vanishes outside $\pi_1(\mathrm{supp} f)$.

From $(*)$ it follows that if $gy_1 = y_2$

then $g_* \nu_{y_1} = \nu_{y_2}$.

Therefore $\int_{\mathcal{P}_n} f \circ g \, d\mu = \int_{\mathbb{R}^n \setminus \{0\}} \int_{\mathcal{P}_n} f \circ g \, d\nu_y \, d\mathrm{Vol}(y)$

$= \int_{\mathbb{R}^n \setminus \{0\}} \int_{\mathcal{P}_n} f \, dg_* \nu_y \, d\mathrm{Vol}(y)$

$= \int_{\mathbb{R}^n \setminus \{0\}} \int_{\mathcal{P}_n} f \, d\nu_{gy} \, d\mathrm{Vol}(y) =$

$= \int_{\mathbb{R}^n \setminus \{0\}} \int_{\mathcal{P}_n} f \, d\nu_x \, d\mathrm{Vol}(x) = \int_{\mathcal{P}_n} f \, d\mu.$

Hence μ is a multiple of $m_{\mathcal{P}_n}$.

So for $f \in L^1(\mathbb{R}^n, \mathrm{Vol})$, $\bar{f} = f \circ \pi_1$,

$$\int_{\mathcal{P}_n} \bar{f} \, d\nu_x = f(x) \Longrightarrow \int_{\mathcal{P}_n} \bar{f} \, d\mu = \int_{\mathbb{R}^n} f \, d\mathrm{Vol}$$

So $\exists\, c_n'' > 0$ s.t.

$$(***) \quad \int_{\mathbb{R}^n} f \, d\mathrm{Vol} = c_n'' \int_{\mathcal{P}_n} \bar{f} \, dm_{\mathcal{P}_n}$$

By Lemma 1, $\exists\, c_n''' > 0$, $\forall h \in L^1(\mathcal{P}_n, m_{\mathcal{P}_n})$

$$(****) \quad \int_{\mathcal{P}_n} h \, dm_{\mathcal{P}_n} = c_n''' \int_{\mathcal{X}_n} h^* \, dm_{\mathcal{X}_n}.$$

Putting (***) and (****) together, get (SSF).